AD 407 165

AFOSR 4543

63-4-1

8

CORNELL UNIVERSITY

Center for Radiophysics and Space Research

ITHACA, N. Y.

CENTER FOR RADIOPHYSICS AND SPACE RESEARCH
CORNELL UNIVERSITY
ITHACA, NEW YORK

February, 1963

CRSR 128

SECULAR CHANGES IN THE SOLAR SYSTEM

Peter Goldreich

ACKNOWLEDGMENTS

The three problems considered in this report were suggested by my adviser, Professor Thomas Gold. Furthermore, many of the essential ideas necessary for their solutions were contributed by him. His countless suggestions and criticisms guided my worked at all stages.

To Gordon MacDonald I owe thanks for a lucid explanation of the latest data on tidal dissipation in the earth. I also wish to thank Professors Martin Harwit, Michael Laird, and Philip Morrison for some valuable suggestions in connection with portions of this report.

I would like to acknowledge the support I received from a National Science Foundation Fellowship during the period I was writing this report. In addition, I would also like to acknowledge the support of the Air Force Office of Scientific Research under Grant AFOSR-62-191.

Finally, I am indebted to Mrs. Agnes Echandi for the typing of the manuscript.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

On the Inclination of Satellite Orbits About an Oblate Precessing Planet

Introduction

It is shown that satellite orbits about an oblate precessing planet will maintain a constant inclination to the planet's equator under a certain condition. This condition is that the motions of the satellite's pericenter and of its ascending node on the equator plane are rapid when compared to the precession of the planet. This is the case for both satellites of Mars and explains why it is not just a coincidence that both of these satellites have small (<2°) inclinations to the equator of Mars at the present time.

1. It is well-known that in the absence of satellites, an oblate spinning planet whose equator is inclined to its orbit plane will precess about the normal to its orbit plane with period given by $T=\frac{2\,(P')^2\,C\sec\phi}{3\,P\,(C-A)}$ [1] (1-1). P is the period of rotation of the planet and P′ is its period of revolution about the sun. C and A are the planet's moments of inertia about an axis perpendicular to the equator plane and an axis in the equator plane respectively. For simplicity we assume axial symmetry for the planet. ϕ is the angle

between the planet's orbit and equator planes, also known as the obliquity. For Mars formula (1-1) gives us $T=2.17\times10^5$ years where $\frac{C-A}{C}$ is taken from satellite data.[2]

If ϕ is the angle between the planet's equatorial and orbit planes then the inclination, relative to the equator, of a satellite orbit which is fixed in space will vary as the planet precesses. If δ is the angle between the satellite and planet orbit planes then the inclination would vary by 2ϕ for $\delta \geq \phi$ and by 2δ for $\delta \leq \phi$. For Mars $\phi=25^\circ\ 12'$. However, Phobos and Deimos are inclined to Mars' equator by less than two degrees. Similar results hold for the inner satellites of Jupiter, Saturn, Uranus, and Neptune. Since these planets are all precessing it is evident that they must drag the orbit planes of their satellites around with them. Otherwise the low inclinations of these satellites would amount to an unbelievable coincidence. Two possibilities suggest themselves since the equator of a planet is defined both by the spin axis of the planet and by the equatorial bulge that this spin produces. In the first place, the spin can affect the inclination through tides raised on the planet by the satellite. These tides transfer spin angular momentum into orbital angular momentum of the satellite or vice versa depending on whether the orbital period or the spin period is longer. However, upper bounds can be put on these tidal effects and they definitely rule out this possibility as an explanation of

the inclinations of the satellites. The second possibility is directly related to the equatorial bulge of the planet. It is well known that the major effect of the oblateness of a planet on its satellites' orbits is to produce a secular motion of the pericenters and of the nodes. In this paper it is shown that if these motions are sufficiently rapid then the inclination of the satellite's orbit relative to the planet's equator is unchanged by the planet's precession. It is then apparent that in this case if satellites are brought into their planets' equator planes (either by being formed there or by some other means) then their inclinations to this plane will remain constant as the planet precesses.

2. The notation which is followed in this paper is set out below.

Ω is the longitude of the ascending node.

i is the orbital inclination relative to the planet's equator.

ω is the longitude of the pericenter.

a is the semi-major axis.

e is the eccentricity.

f is the true anomaly.

M is the mean anomaly. $M=n(t-t_0)$

n is the mean motion. $n=\frac{2\pi}{T}$ where

T is the period of the satellite.

n' is the mean motion of the planet about the sun.

T' is the planet's period about the sun.

s is the spin angular velocity of the planet.

R is the radius of the planet.

M is the mass of the planet.

Explanation of Fig.(1-2).

The two coordinate systems which will be used are drawn in Fig.(1-2). O is their common origin.

The X, Y, Z system is an inertial system with the X Y plane coinciding with the planet's orbital plane. $\hat{I}$, $\hat{J}$, $\hat{K}$ are the usual unit vectors.

The x, y, z set is one in which the x y plane is the planet's equatorial plane. The x axis lies in the X Y plane and makes an angle of α with the X axis. Hence, $\mu = \dot{\alpha}$ is the angular velocity of precession and ϕ is the planet's obliquity. $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit coordinate vectors of this system. $\vec{\mu}=\mu\hat{K}=\mu(\sin\phi\ \hat{j}+\cos\phi\ \hat{k})$.

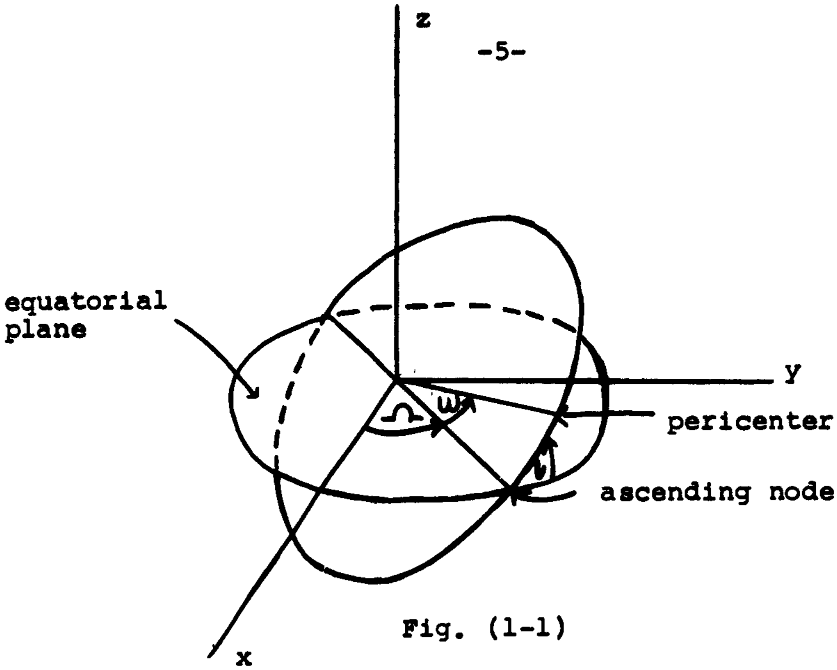
z
equatorial
plane
y
pericenter
ascending node
x
Ω
ω
i

Fig. (1-1)

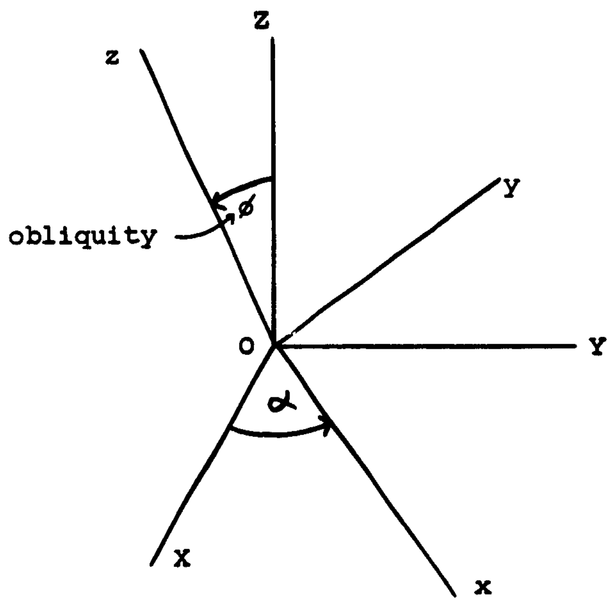
Z
z
obliquity
ϕ
y
O
Y
α
X
x

Fig. (1-2)

Let r be the radius vector to the satellite from O.

$$r = |\vec{r}| \qquad \hat{r} = \frac{\vec{r}}{r} \qquad r = \frac{a(1-e^2)}{1+e\cos f} \qquad (1\text{-}2).$$

Let $\hat{v}$ be a unit vector in the orbit plane orthogonal to $\hat{r}$ and in the direction of increasing f.

Finally, let $\hat{w} = \hat{r} \times \hat{v}$.

We will need the matrix which relates $\hat{r}$, $\hat{v}$, $\hat{w}$ to $\hat{i}$, $\hat{j}$, $\hat{k}$. This is just the Euler matrix and can be written as follows:

$$\begin{bmatrix} \hat{r} \\ \hat{v} \\ \hat{w} \end{bmatrix} = \begin{pmatrix} \\ A \\ \\ \end{pmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

where the matrix A is given below

$$\begin{bmatrix} \cos u \cos\Omega - \sin u \sin\Omega \cos i, & \cos u \sin\Omega + \sin u \cos\Omega \cos i, & \sin u \sin i \\ -\sin u \cos\Omega - \cos u \sin\Omega \cos i, & -\sin u \sin\Omega + \cos u \cos\Omega \cos i, & \cos u \sin i \\ \sin i \sin\Omega, & -\sin i \cos\Omega, & \cos i \end{bmatrix}$$

We have set $u = \omega + f$ in the above. (1-3).

3. The equation for $\frac{di}{dt}$ is given in terms of the perturbation force per unit mass $\vec{F}$ as follows:[3]

$$\frac{di}{dt} = \frac{1}{na(1-e^2)^{1/2}} \left(\frac{r}{a}\right) \cos(\omega + f) \left[\vec{F} \cdot \hat{w}\right] \qquad (1\text{-}4).$$

In the x, y, z frame $\vec{F}$ is made up of two parts. The first is due to the planet's oblateness and the second is the inertial force due to the planet's precession. The oblateness has been treated by many authors including Kozai[4] whose work we

shall refer to for results at a later stage. In the present paper a development of the inertial force will be carried out.

Let $\vec{F}_I$ denote the inertial force and $\vec{F}_O$ the force due to the oblateness.

Hence, $\vec{F}=\vec{F}_O+\vec{F}_I$.

$\vec{F}_I$ has the familiar form

$\vec{F}_I=-2\vec{\mu}\times\dot{\vec{r}}-\dot{\vec{\mu}}\times\vec{r}-\vec{\mu}\times(\vec{\mu}\times\vec{r})$. [5]

The dot denotes differentiation with respect to time in the precessing x, y, z coordinate system. $\vec{\mu}$ is the precession angular velocity and is expressed in terms of the $\hat{i}$, $\hat{j}$, $\hat{k}$ unit vectors as $\vec{\mu}=\mu(\sin\phi\ \hat{j}+\cos\phi\ \hat{k})$. Finally, since we are considering uniform precession, we can set $\dot{\vec{\mu}}=0$ and we get

$\vec{F}_I=-2\vec{\mu}\times\dot{\vec{r}}-\vec{\mu}\times(\vec{\mu}\times\vec{r})$.

The next step is to express $r\cos(\omega+f)[\vec{F}_I\cdot\hat{w}]$ in terms of trigonometric functions of the true anomaly f. We shall treat the "centrifugal force" $-\vec{\mu}\times(\vec{\mu}\times\vec{r})$ first.

$\vec{\mu}\times(\vec{\mu}\times\vec{r})\cdot\hat{w}=(\vec{\mu}\cdot\vec{r})(\vec{\mu}\cdot\hat{w})-(\vec{\mu}\cdot\vec{\mu})(\vec{r}\cdot\hat{w})$

but $(\vec{r}\cdot\hat{w})=0$. Hence, we have $r\cos(\omega+f)[\vec{\mu}\times(\vec{\mu}\times\vec{r})\cdot\hat{w}]=r\cos(\omega+f)(\vec{\mu}\cdot\vec{r})(\vec{\mu}\cdot\hat{w})$. Using the transformation matrix (1-3) we can express $\vec{r}$ and $\hat{w}$ in terms of $\hat{i}$, $\hat{j}$, $\hat{k}$.

A simple algebraic computation then yields:

$$-\frac{4}{\vec{\mu}^2 r}\cos(\omega+f)[\vec{\mu}\times(\vec{\mu}\times\vec{r})\cdot\hat{w}] =$$

$$+\sin^2\phi\begin{bmatrix}(2\cos^2 f\cos^2\omega+2\sin^2 f\sin^2\omega-\sin 2f\sin 2\omega)\sin 2\Omega\sin i\\(\sin 2f\cos 2\omega+\cos 2f\sin 2\omega)\cos^2\Omega\sin 2i\end{bmatrix}$$

$$-\cos^2\phi\left[(\sin 2f\cos 2\omega+\cos 2f\sin 2\omega)\sin 2i\right]$$

$$-\sin 2\phi\begin{bmatrix}(2\cos^2 f\cos^2\omega+2\sin^2 f\sin^2\omega-\sin 2f\sin 2\omega)\sin\Omega\cos i\\(\sin 2f\cos 2\omega+\cos 2f\sin 2\omega)\cos\Omega\cos 2i\end{bmatrix}$$

(1-5)

Next we shall treat the "Coriolis force"

$$2r\cos(\omega+f)\left[(\vec{\mu}\times\dot{\vec{r}})\cdot\hat{w}\right]=2r\cos(\omega+f)[\vec{\mu}\cdot(\dot{\vec{r}}\times\hat{w})]$$

$$2r\cos(\omega+f)\,\vec{\mu}\cdot(\dot{r}\hat{r}+r\dot{f}\hat{\vartheta})\times\hat{w}=2r\cos(\omega+f)\,\vec{\mu}\cdot\left(r\dot{f}\hat{r}-\frac{er^2\sin f\dot{f}\hat{\vartheta}}{a(1-e^2)}\right)$$

Where I have made use of

$$\dot{r}=\dot{\left(\frac{a(1-e^2)}{1+e\cos f}\right)}=\frac{er^2\sin f\dot{f}}{a(1-e^2)} \qquad (1\text{-}6)$$

Using $\vec{\mu}=\mu(\sin\phi\hat{j}+\cos\phi\hat{k})$ we get after an elementary computation

$$2r\cos(\omega+f)\left[(\vec{\mu}\times\dot{\vec{r}})\cdot\hat{w}\right]=$$

$$\frac{\sin\phi(r^3\dot{f})}{a(1-e^2)}\begin{bmatrix}\begin{Bmatrix}(2\cos^2 f\cos^2\omega+2\sin^2 f\sin^2\omega-\sin 2f\sin 2\omega)\sin\Omega\\(\sin 2f\cos 2\omega+\cos 2f\cos 2\ \ )\end{Bmatrix}(1+e\cos f)\\\begin{Bmatrix}(\sin 2f\cos 2\omega+\cos 2f\cos 2\omega)\sin\Omega\\(\sin 2f\sin 2\omega-\cos^2 f\cos^2\omega-\sin^2 f\sin^2\omega)\cos\Omega\cos i\end{Bmatrix}e\sin f\end{bmatrix}$$

$$\frac{\cos\phi(r^3\dot{f})}{a(1-e^2)}\begin{bmatrix}\{(\sin 2f\cos 2\omega+\cos 2f\cos 2\omega)\sin i\}(1+e\cos f)\\\{(\sin 2f\sin 2\omega-2\cos^2 f\cos^2\omega-2\sin^2 f\sin^2\omega)\sin i\}e\sin f\end{bmatrix}$$

(1-7).

4. At this stage we will take the time average of $\frac{di}{dt}$ due to $\vec{F}_I$ over a single orbit of the satellite. In order to carry

out this average we shall need a formula relating f to M. This formula follows directly from the equations of motion.

i) $\ddot{r}-r\dot{\theta}^2=-\frac{G\ M_P}{r}$ ii) $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0$

ii) implies $r^2\frac{d\theta}{dt}=h=\text{constant}$. Since f differs from θ by a constant angle, $\frac{df}{dt}=\frac{h}{r^2}$. Using formula (1-6) we have

$\frac{er^2\sin f\dot{f}}{a(1-e^2)}=\frac{eh\sin f}{a(1-e^2)}=\dot{r}$

$$\ddot{r}=\frac{eh\cos f\dot{f}}{a(1-e^2)}=\frac{eh^2\cos f}{a(1-e^2)r^2}$$

$$r\dot{\theta}^2=h^2$$

hence i) gives $\frac{h^2}{r^3}\ \frac{-eh^2\cos f}{a(1-e^2)r^2}=\frac{G\ M_P}{r^2}$

$h=G\ M_P a(1-e^2)$ is the result we are after.

$\frac{df}{dM}=\frac{df}{dt}\ \frac{dt}{dM}=\frac{1}{n}\ \frac{df}{dt}=\frac{h}{nr^2}=\frac{\sqrt{G\ M_P a}\sqrt{1-e^2}}{n\ r^2}$

$\frac{df}{dM}=\frac{a^2\sqrt{1-e^2}}{r^2}$ using $G\ M_P=n^2a^3$. (1-8).

We shall denote time averages by a bar over the averaged quantity.

Thus $\bar{F}=\frac{1}{2\pi}\int_0^{2\pi}FdM.$ (1-9).

In performing the averages we can change the integration variable to f thus obtaining $F=\frac{1}{2\pi\sqrt{1-e^2}}\int_0^{2\pi}\left(\frac{r}{a}\right)^2Fdf$ (1-10).

Using this formula for computing averages and noting that r is a symmetric function of f about $f=\pi$, we can immediately observe that all terms involving an odd power of sinf will

give zero upon averaging. This is the only property of these averages that we shall need aside from the obvious fact that they are functions of a and e only. Dropping the terms which average to zero we can write:

$$\frac{di}{dt}=\frac{1}{na^2(1-e^2)^{1/2}}\left[-\frac{\mu^2}{4}\sin^2\phi\left\{(2\overline{r^2\sin^2 f}\sin^2\omega+2\overline{r^2\cos^2 f}\cos^2\omega)\sin 2\Omega\sin i\right.\right.$$

$$\left.+(\overline{r^2\cos 2f}\sin 2\omega\cos^2\omega\sin 2i)\right\}+\frac{\mu^2}{4}\cos^2\phi\left\{\overline{r^2\cos 2f}\sin 2\omega\sin 2i\right\}$$

$$+\frac{\mu^2}{4}\sin 2\phi\left\{(2\overline{r^2\cos^2 f}\cos^2\omega+2\overline{r^2\sin^2 f}\sin^2\omega)\sin\Omega\cos i\right.$$

$$\left.+(\overline{r^2\cos 2f}\sin 2\omega\cos\Omega\cos 2i)\right\}$$

$$\frac{\mu\sin\phi}{a(1-e^2)}\left\{2\overline{r^2 f\cos^2 f(1+e\cos f)}\cos^2\omega\sin\Omega\right.$$

$$2\overline{r^3 f\sin^2 f(1+e\cos f)}\sin^2\omega\sin\Omega+\overline{r^3 f\cos 2f(1+e\cos f)}\cos 2\omega$$

$$\left.e\overline{r^3 f\sin 2f\sin f}\sin 2\omega\cos\Omega\cos i\right\}+\frac{\mu\cos\phi}{a(1-e^2)}\left\{e\overline{r^3 f\sin 2f\sin f}\sin 2\omega\sin i\right.$$

$$\left.\left.\overline{r^3 f\cos 2f(1+e\cos f)}\cos 2\omega\sin i\right\}\right]$$

(1-11)

5. At this stage we must refer to the results of the paper by Kozai. To a first approximation the potential due to an oblate planet can be expressed as

$V=\frac{G\,M_p}{r}+\frac{G\,M_p A_2}{r}(\frac{1}{3}-\sin^2\lambda)$ where λ is the angle of latitude measured from the equator. For an axially symmetric (about a polar axis) planet $A_2=\frac{3(C-A)}{2\,M_p}$ [6] (1-12)

where C is the moment of inertia about the polar axis and A is the moment of inertia about an equatorial axis. In this approximation the disturbing potential

$R=\frac{G\,M_p A_2}{r^3}(\frac{1}{3}-\sin^2\lambda)$. In Kozai's paper it is shown that of the six orbital elements a, e, i, ω, Ω, M only the last three undergo secular (non-periodic) perturbations. Furthermore, the secular perturbation of M is a trivial one and can be considered as defining a new or "renormalized" mean motion. The secular perturbations in ω and Ω can be expressed in terms of n and A_2 as follows:

$$\omega=\omega_0+\frac{2nA_2}{a^2(1-e^2)^2}\left[1-\frac{5}{4}\sin^2 i\right]t$$

$$\Omega=\Omega_0-\frac{A_2 n\cos i\,t}{a^2(1-e^2)^2} \qquad (1\text{-}13).$$

6. The last step in the proof involves averaging the expression for $\overline{\frac{di}{dt}}$ over a period of time which is long compared to the periods of revolution of the pericenter and node. It

is here that the assumption of a slow precession with respect to these periods is employed. This assumption allows us to treat a, e and i as constants in this average and only consider the motions of ω and Ω since the oblateness only produces secular variations of these latter two elements as mentioned in section 5.

If we examine the equation for $\overline{\frac{di}{dt}}$ under the conditions above, namely, that ω and Ω are linear functions of the time and a, e and i are constants, we see that all terms on the right-hand side have zero time average providing that we average over a sufficiently long period of time.

For the most interesting case of small inclination we have $\frac{d\omega}{dt} \approx -\frac{2d\Omega}{dt}$ and it is seen by inspection that upon averaging over a time equal to $T = \frac{2\pi}{\dot{\Omega}} = \frac{2\pi a^2(1-e^2)^2}{nA_2}$ gives us $\frac{1}{T}\int_0^T \overline{\frac{di}{dt}}\, dt = 0$ (1-14). In special cases such as $\sin^2 i \approx \frac{4}{5}$ we have $\frac{d\omega}{dt} \approx 0$ and in this case terms which are independent of Ω must be averaged over much longer periods of time before they give zero. Other pathological cases occur when $\frac{d\omega}{dt} \approx -\frac{d\Omega}{dt}$ and in similar circumstances. These cases give rise to products of trigonometric functions whose arguments change with time at approximately the same rates and this also increases the time interval over which these terms average to zero.

7. The condition under which the inclination is essentially constant, except for small periodic terms, is expressed in a more illuminating form in this paragraph. From formula (1-1) we have

$$\mu = \frac{3n'^2}{2s}\frac{(C-A)}{C}\cos\phi \qquad \text{Also } \dot{\Omega} = \frac{d\Omega}{dt} = \frac{A_2 n}{a^2(1-e^2)^2}$$

when $\sin i \ll 1$. Using $A_2 = \frac{3}{2}\frac{(C-A)}{M_\rho}$ we get

$$\frac{\dot{\Omega}}{\mu} = \frac{nsC\sec\phi}{n'^2 M_\rho a^2(1-e^2)^2} = \frac{3}{5}\left(\frac{ns}{n'^2}\right)\frac{\sec\phi}{(1-e^2)}\frac{R}{a} \qquad (1\text{-}15).$$

Now the condition for constant inclination becomes

$$\frac{\dot{\Omega}}{\mu} \gg 1 \qquad \text{or} \qquad \frac{3}{5}\left(\frac{ns}{n'^2}\right)\frac{\sec\phi}{(1-e^2)^2}\left(\frac{R}{a}\right)^2 \gg 1$$

For Mars we have $\frac{\dot{\Omega}}{\mu} = 1.25\times10^6$ for Phobos while $\frac{\dot{\Omega}}{\mu} = 5.0\times10^4$ for Deimos.

We see from the magnitudes of $\frac{\dot{\Omega}}{\mu}$ for Phobos and Deimos that these satellites should undergo no change in inclination due to the precession of Mars. At present the orbit of Phobos is inclined by 1.8° and that of Deimos by 1.4° to the equator plane of Mars. Since the obliquity of Mars is 25° 12' it would be a remarkable coincidence if the inclinations of these satellites varied by about 50° (see section 1) and just at that time we came to observe them they both were inclined by less than 2° to Mars' equator. In light of the previous discussion we can see that this is not a coincidence at all since the inclinations of these satellites do not vary as Mars precesses.

8. In this section a more refined calculation on orbits of very low eccentricity $e \ll 1$ and small inclination $\sin i \ll 1$ is carried out. The amplitudes of the periodic terms in the inclination are evaluated. From formula (1-11) we have, setting $e=o$, and working to first order in $\sin i$

$$\frac{dI}{dt}=\frac{\mu^2}{4n}(-\sin^2\phi\sin 2\Omega\sin i+\sin 2\phi\sin\Omega)+\sin\phi\sin\Omega$$

Since $\frac{\mu}{n} \ll 1$ for all satellites and planets in the solar system we can approximate $\frac{dI}{dt}=\mu\sin\phi\sin\Omega$ (1-16). Using equation (1-13) we get

$$\Omega=\Omega_o-\frac{A_2 n}{a^2}=\Omega_o-\frac{3(C-A)n}{2Ma^2}=\Omega_o-\frac{9(C-A)}{10\ C}\ \frac{R^2 n}{a^2}$$

Integrating $\frac{dI}{dt}=\mu\sin\phi\sin\Omega$ yields $I=-\frac{\mu}{\Omega}\sin\phi\cos\Omega+I_o$ where I_o is the constant part of I. Finally, substituting for μ and Λ we get $I=\left[\frac{3n'^2}{5\Omega n}\left(\frac{a}{R}\right)^2\cos\phi\right]\cos\Omega+I_o$ (1-17).

For the satellites of Mars $I=3.45\times10^{-6}\cos\Omega+I_o$ for Phobos and $I=1.38\times10^{-5}\cos\Omega+I_o$ for Deimos or $I=.71''\cos\Omega+I_o$ for Phobos and $I=2.84''\cos\Omega+I_o$ for Deimos.

9. Similar results hold for Jupiter, Saturn, Uranus and Neptune. In all cases the major satellites of these planets are close enough to their planets to maintain fixed inclinations with respect to the equator of the planet.

Some of the outer satellites of the major planets, as well as the Moon, have inclinations which remain constant

with respect to their planet's orbit planes. In all these cases, except that of the Moon, the planets could maintain constant inclinations of the satellites with respect to their equators if no forces other than those of the planet acted on these satellites. However, the sun produces significant perturbations of the elements of these outer satellites. In particular, if we can neglect the oblateness of a planet then the sun produces a motion of the node of a satellite. This motion is uniform on the planet's orbit plane. This is the major motion of the node for some outer satellites and makes inclinations constant with respect to the planetary orbit planes even though the planets are oblate and precessing.

It should be pointed out that the calculations for these major planet-satellite systems are not quite as elementary as those for Mars. This is for two reasons. In the first place, mutual interactions between the satellites produce additional motions of their nodes and pericenters which may be comparable to those produced by the planets' oblateness. Secondly, the torque on the planet-satellite system due to the sun may include significant contributions from couples acting on the satellites in addition to the torque acting on the planet itself. For this reason the formula (1-1) is not always applicable.

Finally, in the Earth-Moon case it is well-known that the Moon produces a greater torque on the Earth than the sun does so (1-1) does not apply here without modification either.

On the Eccentricity of Satellite Orbits in the Solar System

Introduction

In this paper the secular changes in the eccentricities of satellite orbits in the solar system are investigated. Two mechanisms which affect the eccentricities are considered. One of them is the tide raised on the planet by the satellite, which has been the subject of discussion in the past; the other is the tide raised on the satellite by the planet. It is seen that cases arise in the solar system in which each of these tide's effect on eccentricity is dominant.

1. Darwin[1] (1909), Groves[2] (1960) and Jeffreys[3] (1961) have given arguments to show that in most cases the tide raised by a satellite on a planet tends to increase the eccentricity of the satellite's orbit. If we look at the values of the eccentricities which arise in the orbits of the inner satellites of planets, we see that they range down as low as 10^{-4} for Tethys. Since it seems hard to imagine any process of formation of satellites which could produce initial values of eccentricity as low or lower than these, it would appear necessary to look for some mechanism that could produce a secular decrease of eccentricity which

could rival in magnitude the secular change due to the tides raised on the planets. Such a mechanism was proposed by Urey[4] (1958) in the form of tidal working in the satellite due to tides raised by the planet. If we only consider the case where the satellite always presents the same face to the planet (this is the only case which is observed in the solar system and includes the greater satellites of Jupiter and Saturn as well as the Moon. Jeffreys[5] 1952) then it is easy to see why the tide raised on a satellite tends to decrease its eccentricity. The eccentricity $e=\sqrt{1+\frac{2\ E\ L^2}{M_s^3 M_p^2 G}}$ (2-1) where E is the energy of the orbit, L is the angular momentum and M_p and M_s are the planet and satellite masses. If the satellite is not spinning, then the tide raised on it can only produce a radial perturbation force. This means that L is not changed by the tide. Since any energy dissipation in the satellite decreases E and since we have $E<0$, $0<e<1$ and L constant, we find that e is decreasing also. If $e\neq 0$ such dissipation must take place since the height of the tide will vary with the oscillation in distance between the satellite and planet.

In this paper it will be shown that insofar as their effects on eccentricity are concerned, the tides on the satellites are probably more important than tides on the planets in all cases where tidal effects might be significant except for Phobos, Deimos and possibly the Moon and Jupiter

V. In the cases of Phobos, Deimos and Jupiter V, however, the tides raised in their planets tend to secularly decrease their eccentricities anyway so it is not surprising to find that the orbits of these satellites have low values of eccentricity.

2. In this section the following idealized case of a planet with a single satellite will be considered. The assumptions are as follows:

i) The mass of the satellite is neglected in comparison with that of the planet.

ii) The inclination of the satellite's orbit plane to the planet's equator is taken as zero.

iii) The planet and satellite are both considered to be homogeneous incompressible spheres which can be characterized by two parameters, μ and Q. μ is the rigidity and Q the specific dissipation function.

$$Q = \frac{2\pi E^*}{\oint \frac{dE}{dt}\, dt} \qquad (2-2)$$

where E^* is the peak energy stored in the system during a cycle and $\oint \frac{dE}{dt}\, dt$ is the energy dissipated over a complete cycle. Q will in general vary with the frequency and amplitude of the tide and the size of the sphere in addition to its composition.

iv) Finally, I will work only to first order in eccentricity in the interest of simplicity.

The idealized problem that I have set up has been solved neglecting tides raised on the satellite by Darwin[6] (1908) and reviewed by Jeffreys[7] (1961). The following will be a summary of Jeffreys' paper with several corrections of misprints. Jeffreys' notation will not be followed; instead we shall use the following:

a=semi-major axis of the satellite's orbit.

e=eccentricity of the satellite's orbit.

G=gravitational constant.

n=satellite's mean motion=$\frac{2\pi}{T}$ where

T=satellite's period.

Quantities pertaining to the planet will have a subscript p and those pertaining to the satellite will have a subscript s.

M=mass of sphere.

R=radius of sphere.

ρ=density of sphere.

g=surface gravity of sphere.

μ=rigidity of sphere.

$Q(\nu)$=Q as a function of frequency ν.

ω=angular rotation velocity of sphere.

$2\varepsilon_i$ $i=1\rightarrow3$ are phase lags in the periodic tides raised by the tidal potential.

These phase lags arise as follows: the tidal potential acting is written as a sum of periodic terms with different

frequencies. The response of the tide to any periodic component of the potential will be in phase with the potential only if no energy is dissipated by the tide. If the tide dissipates energy, then its phase will lag that of the potential (at least on the average). $2\varepsilon_0$, $2\varepsilon_1$, $2\varepsilon_2$, and $2\varepsilon_3$ are the lags of the tides with frequencies $2\omega-2n$, $2\omega-3n$, $2\omega-n$, and $\frac{3}{2}n$ respectively.

Finally, using the notation set up above, we can state Darwin's and Jeffreys' result.

$$\overline{\frac{de}{dt}}_p = \frac{-6}{5}(G\,M_p)^{1/2} R_p\, H_p \frac{e}{a^{7/2}}\left(\varepsilon_{op} - \frac{49}{4}\varepsilon_{1p} + \frac{1}{4}\varepsilon_{2p} + \frac{3}{2}\varepsilon_{3p}\right) \qquad (2\text{-}3)$$

where $\frac{\bar{d}}{dt} = \frac{\bar{d}}{dt}_p + \frac{\bar{d}}{dt}_s$. $\frac{\bar{d}}{dt}_p$ denotes secular rate of change due to tides on the planet and $\frac{\bar{d}}{dt}_s$ denotes secular rate of change due to tides on the satellite.

The formulas for λ and H in Jeffreys' paper, however, contain several mistakes so they will be corrected here. The elastic tide raised by a potential $U = k_2 r^2 S_2$ (2-4) (where S_2 is a spherical harmonic of order two and r is the distance from the center of the planet) is equal to $\lambda R_p\, S_2$ (2-5) at the surface of the sphere where

$$\lambda = \frac{5\rho_p R_p^2\, k_2}{(19\mu_p + 2g_p\rho_p R_p)} \text{ and } k_2 = \frac{3}{4}\frac{G\,M_s}{a^4} \quad (2\text{-}6).$$

If we write $\lambda R_p = H_p$ then we find that the surface inequality of the planet produces an external potential $U_{1p}(r) = \frac{3}{5}\frac{G\,M_p R_p H_p S_2}{r^3}$.

Using the correct expression for H we get:

$$\overline{\frac{de}{dt}}_p = \frac{9(G\ M_p)^{1/2}(G\ M_s)R_p^4\rho_p e}{2a^{13/2}(19\mu_p+2g_p\rho_p\ R_p)}\left[\varepsilon_{0p} - \frac{49}{4}\varepsilon_{1p}+\frac{1}{4}\varepsilon_{2p}+\frac{3}{2}\varepsilon_{3p}\right] \quad (2\text{-}7)$$

If we next consider the tides raised on the satellite we can modify the above argument slightly to get $\overline{\frac{de}{dt}}_s$. We must first realize that $\omega_s=n$. Hence, $2\omega_s-2n=0$, $2\omega_s-3n=-n$, $2\omega_s-n=n$ and as a consequence $\varepsilon_{0s}=0$, and $\varepsilon_{1s}=-\varepsilon_{2s}$. Furthermore, U_{1p} above is the potential energy per unit satellite mass due to tides on the planet. $U_{1s}=\frac{3G\ M_s\ R_s\ H_s\ S_2}{r^3}$ is then the potential energy per unit planet mass due to tides on the satellite. To change this to potential energy per unit satellite mass, we must multiply U_{1s} by $\frac{M_p}{M_s}$. Using the preceding results we have from Jeffreys' formula that

$$\overline{\frac{de}{dt}}_s = -\frac{9\ M_p(G\ M_p)^{1/2}(G\ M_s)R_s^4\rho_s e}{2\ M_s a^{13/2}(19\mu_s+2g_s\rho_s R_s)}\left[\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s}\right] \qquad \text{or}$$

$$\overline{\frac{de}{dt}}_s = -\frac{9\ (G\ M_p)^{3/2}R_s^4\rho_s e}{2a^{13/2}(19\mu_s+2g_s\rho_s R_s)}\left[\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s}\right] \quad (2\text{-}8)$$

3. Before we can compare the magnitudes of $\overline{\frac{de}{dt}}_p$ and $\overline{\frac{de}{dt}}_s$, we must relate the phase lags, $2\varepsilon_i$, to the specific dissipation function Q. This is accomplished in the following manner. Let W be the work done on one of our homogeneous, incompressible spheres by a body force $\vec{f}(\vec{r})$ which is derivable from a potential $U(\vec{r})$. Then

$\frac{dW}{dt} = \int \rho \vec{v}(\vec{r}) \cdot \vec{f}(\vec{r}) d^3r$ (2-9) where $\vec{v}(\vec{r})$ is the velocity of the material at point $\vec{r}$. Using the equation of continuity we get $\rho\vec{v}\cdot\nabla U = \nabla\cdot(\rho\vec{v}U) - U\nabla\cdot(\rho\vec{v}) = \nabla\cdot(\rho\vec{v}U) + \frac{\partial\rho}{\partial t} U$. Since $\frac{\partial\rho}{\partial t} = 0$, we have $\frac{dW}{dt} = \int \rho U \vec{v}\cdot\hat{n} ds$ (2-10) where $\hat{n}$ is the outward normal to the surface of the sphere. If $U \propto \cos\nu t$ and the surface inequality is $\propto \cos(\nu t - 2\varepsilon)$ then $\vec{v}\cdot\hat{n} \propto \sin(\nu t - 2\varepsilon)$. Hence, we get $\frac{dW}{dt} = K\cos\nu t \sin(\nu t - 2\varepsilon)$. A simple integration gives to first order in ε.

$$\oint \frac{dW}{dt} dt = -\frac{K\pi}{\nu}\sin\varepsilon = \oint \frac{dE}{dt} dt$$

$$E^* = \int_0^{\pi/2\nu} \frac{dW}{dt} dt = \frac{K}{2\nu}\cos 2\varepsilon$$ Using (2-2) we get

$$Q = \frac{2\pi E^*}{\oint \frac{dE}{dt} dt} = \frac{1}{\tan 2\varepsilon}$$ or for large Q $2\varepsilon = \frac{1}{Q}$. (2-11)

Incidentally, this calculation tells us that $\varepsilon(\nu)$ has the same sign as ν since $\oint \frac{dE}{dt} dt < 0$

4. Next I will show how Q compares for two spheres of the same material but of different size. This behavior is important for explaining why the inner satellites of the large planets have circular orbits. (e.g. Saturn I→V).

Qualitatively one would expect Q to increase with the size of the body for the following reason. The energy dissipated, per unit volume, in a cycle, will depend only on the square of the strains since this frictional energy

dissipation is a local phenomenon. The peak energy stored per unit volume, on the other hand, increases for a fixed strain with the size of the body since the stresses are increased over the purely elastic ones by the self-gravitation of the sphere. Hence, Q increases as the bodies self-gravity becomes more important than its elasticity. The relative importance of elasticity and self-gravity of the sphere enters into the formula for the tidal surface inequality in the denominator $(19\mu+2g\rho R)$. To examine the quantitative dependence of Q on size, we proceed as follows. From formulas (2-4), (2-5), (2-6) and (2-10) we have the energy dissipated in a cycle for each periodic component of the tide equals $E_i^* = \oint \frac{dW_i}{dt}\, dt = \frac{C_i R^7 \varepsilon_i}{(19\mu+2g\rho R)}$ where C_i is independent of μ and R. However, the energy dissipated must be proportional to the square of the strain as mentioned previously. Therefore, the energy dissipated in a cycle equals $\frac{D_i R}{(19\mu+2g\rho R)}$ where D_i is independent of μ and R also. This last formula is derived as follows. Using (2-5) and (2-6) we have the surface strains proportional to $\frac{R^2}{(19\mu+2g\rho R)}$ hence the square of the strain itegrated over the volume of the sphere gives a result proportional to $\frac{R^7}{(19\mu+2g\rho R)}$ where we assume the surface value of strain throughout the sphere as an approximation to get the proportional result. Comparing these expressions we get $\varepsilon_i = \frac{\gamma}{(19\mu+2g\rho R)}$ where γ is

independent of μ and R. This gives $\frac{Q}{Q_0}=1+\frac{2g\rho R}{19\mu}$ (2-12) where Q_0 is the value of Q for a body where self-gravity is negligible. In the case of liquid or gaseous spheres $\mu=0$ and Q_0 is not defined.

5. In this section the two rates of change of e will be compared both in magnitude and in sign.

First I shall deal with the question of sign. $\overline{\frac{de}{dt}}_p$ has the sign of $-(\varepsilon_{0p}-\frac{49}{4}\varepsilon_{1p}+\frac{\varepsilon_{2p}}{4}+\frac{3}{2}\varepsilon_{3p})$. For the Earth, Q and hence ε, varies by less than a factor of four over a range of one cycle per second to one cycle per year.[8] In this case ε_1 must be the dominant term and the sign of $\overline{\frac{de}{dt}}_p$ is the same as the sign of $2\omega-3n$. While this constant behavior of Q with frequency may not be true for all planets (especially not the major ones) it is still likely that the ε_1 term is dominant because of its relatively large coefficient. If this ε_1 term is dominant, we have $\overline{\frac{de}{dt}}_p>0$ for all satellites except Phobos, Deimos, Jupiter V and the retrograde ones.

In the case of tides raised on satellites the sign of $\overline{\frac{de}{dt}}_s$ is the sign of $-(\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s})$ where ε_{2s} and ε_{3s} have the sign of n and therefore are positive. Hence, $\overline{\frac{de}{dt}}_s<0$ for all satellites which keep the same face toward their planets.

Next, we will compare the magnitudes of these two rates of change of e. From formulas (2-4) and (2-8) we get

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} = \left(\frac{R_s}{R_p}\right) \frac{(19\mu_p+2g_p\rho_pR_p)}{(19\mu_s+2g_s\rho_sR_s)} \frac{\left\{\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s}\right\}}{\left\{\varepsilon_{op}-\frac{49}{4}\varepsilon_{1p}+\frac{\varepsilon_{2p}}{4}+\frac{3}{2}\varepsilon_{3p}\right\}} \qquad (2\text{-}13)$$

I will examine this ratio in three limiting cases:

i) $\mu_p \gg 2g_p\rho_pR_p \qquad \mu_s \gg 2g_s\rho_sR_s$

This is the case of small satellite and small planet and yields the result

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} \approx \left(\frac{R_s}{R_p}\right)\left(\frac{\mu_p}{\mu_s}\right) \frac{\left\{25\varepsilon_{2s}+3\varepsilon_{3s}\right\}}{\left\{2\varepsilon_{op}-\frac{49}{2}\varepsilon_{1p}+\frac{\varepsilon_{2p}}{2}+3\varepsilon_{3p}\right\}} \qquad (2\text{-}14)$$

We see that if the satellite is appreciably smaller than the planet and has approximately the same rigidity and specific dissipation function, we get the tides raised on the planet dominating. This case certainly applies to Phobos and Deimos.

ii) $\mu_p \ll 2g_p\rho_pR_p \qquad \mu_s \gg 2g_s\rho_sR_s$

This is the case of large planet and small satellite and we get

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} \simeq \frac{2g_p\rho_pR_s}{19\mu_s} \frac{\left\{\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s}\right\}}{\left\{\varepsilon_{op}-\frac{49}{4}\varepsilon_{1p}+\frac{\varepsilon_{2p}}{4}+\frac{3}{2}\varepsilon_{3p}\right\}} \qquad (2\text{-}15)$$

In this case no general conclusion can be drawn about this ratio.

iii) $\mu_p \ll 2g_p\rho_p R_p$ $\qquad\qquad$ $\mu_s \ll 2g_s\rho_s R_s$

This is the case of large planet and large satellite and gives

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} \simeq \left(\frac{\rho_p}{\rho_s}\right)^2\left(\frac{R_p}{R_s}\right)\frac{\left\{\frac{25}{2}\varepsilon_{2s}+\frac{3}{2}\varepsilon_{3s}\right\}}{\left\{\varepsilon_{op}-\frac{49}{4}\varepsilon_{1p}\ \frac{\varepsilon_{2p}}{4}+\frac{3}{2}\varepsilon_{3p}\right\}} \tag{2-16}$$

In this case we see that the satellite tide wins. This is made even more striking when we remember that if the satellite has the same composition as the planet then

$$\frac{Q_p}{Q_s} = \left(\frac{R_p}{R_s}\right)^2 \text{ so that } \frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} \simeq \left(\frac{R_p}{R_s}\right)^2 \tag{2-17}$$

6. A discussion of the results obtained in the preceeding sections is presented. This discussion is intended to give an explanation of the results obtained so far and to indicate the range of their validity.

The first point dealt with will be the effect that tides raised on the planet have on the eccentricity. Let us consider the case where the period of the planet's rotation is much shorter than the period of the satellite's revolution and the satellite is a direct one (not retrograde). Since the tide raised on the planet is dissipative,

we have a time lag between the applied tidal force and the tidal bulge it raises. Because the day is shorter than the month this time lag means that the tidal bulge precedes the satellite in longitude. The effect of this tidal bulge lead in longitude is to produce a couple between the satellite and planet which adds angular momentum to the satellite's orbit, at the expense of the rotational angular momentum of the planet. This is the well-known tidal couple which is responsible for the secular accelaration of the Moon.

So far our argument has been independent of the eccentricity of the satellite's orbit. Let us next consider what additional complications arise when we take the eccentricity into account and how they feed back to effect the eccentricity which produced them. In order to simplify the picture let us think of a very eccentric satellite orbit. The height of the tide raised depends inversely on the third power of the satellite's distance and the force it produces on the satellite involves four more reciprocal powers of distance. Hence, we have the torque on the satellite decreasing as the sixth power of the satellite's distance from the planet. This steep decrease with distance enables us to approximate the effect of the tidal bulge on the satellite's orbit by an impulse at pericenter. With this approximation the satellite must again pass through the same point at pericenter, since bound orbits in inverse square force fields

are periodic. But, angular momentum has been added to the satellite's orbit; hence, the apocenter distance, and therefore the eccentricity, must have been increased. In fact the angular momentum of the satellite per unit mass is $L = \sqrt{G\ M_p a(1-e^2)} = \sqrt{G\ Mr_p(1+e)}$ where $r_p = a(1-e)$ is the distance to pericenter. If we have $\Delta L>0$ and $\Delta r_p=0$ as discussed above, then $\Delta L=\frac{1}{2}\sqrt{\frac{G\ Mr_p}{(1+e)}}\Delta e$ so Δe is positive. Also from $\Delta r_p=0$ we get $\Delta[a(1-e)] = \Delta a(1-e)-a\Delta e=0$ or $\Delta a=\frac{a\Delta e}{(1-e)}$ so that Δa is positive also, as we would expect. The previous discussion, when modified to hold for smaller values of e, accounts for the tendency of the tide raised on the planet, to increase the satellite's eccentricity.

The considerations presented above are concerned solely with the tidal torque on the satellite. That is, they only make use of the component of the disturbing force which is perpendicular to the satellite's radius vector. In his paper on the Moon's eccentricity, Groves also considered only the tidal torque. It is not surprising then, that he found the Moon's eccentricity could only increase due to tides on the Earth. This neglect of the radial components of the disturbing force renders the above arguments, and those of Groves as well, incomplete. We shall consider how the picture presented up to now is altered by the inclusion of radial forces.

We shall again take an eccentric orbit about our planet. The relevant points are identical with those presented in section 1. There it was shown that the tide raised on the satellite produces only radial perturbation forces and since these cannot change the satellite's angular momentum, but must decrease its energy, they must also decrease its eccentricity. The preceding argument, when applied to the planetary tide, shows us that this tide may decrease as well as increase eccentricity. The details of whether we have decreasing or increasing eccentricity depend on the satellite's revolution period, the planet's rotation period and the amplitude and frequency dependence of $Q(\nu)$.

The applicability of our results to the actual planet-satellite systems extant involves two questionable assumptions.

The first assumption is the neglect of all tides except the solid body ones. It has recently been demonstrated, that in the Earth-Moon system, the ocean tides which in the past were thought to be of major importance are really much less important than the solid body tides. This conclusion would undoubtedly also pertain to Mars. For Jupiter, Saturn, Uranus and Neptune, however, turbulent tides in their atmospheres or possibly in any liquids which may be found on these planets, might be of greater importance than the solid body tides. We can still use the two parameters μ and Q to fit the tides on these planets, although we can no longer

hope to make very good estimates of their frequency and amplitude dependence. The satellites of these planets are almost certainly solid since they are not big enough to have held the heat necessary to keep them liquid and are not receiving enough heat to do this either. Before leaving this question of the composition of the major planets, it should be mentioned that for Jupiter, measurements exist which have been used in the past to calculate a lower bound for Q. This question will be taken up in the following paper where it will be shown that although the measurements may be correct, their interpretation is not.

The second serious approximation we have made in the pretense of a linear superposition for the tides of different frequencies. In developing formula (2-7), Darwin and Jeffreys both wrote the tide raising potential as the sum of periodic potentials. They then proceeded to consider the response of the planet to each of the potentials separately. At first glance this might seem proper since the tidal strains are very small and should add linearly. The stumbling block in this procedure, however, is the amplitude dependence of the specific dissipation function. In the case of the Earth, it has been shown by direct measurement that Q varies by an order of magnitude if we compare the tide of frequency $2\omega-2n$ with the tides of frequencies $2\omega-n$, $2\omega-3n$ and $\frac{3n}{2}$. This is because these latter tides have amplitudes which are smaller than

those of the principal tide (of frequency $2\omega-2n$) by a factor of eccentricity or about .05. It may still appear that we can allow for this amplitude dependence of Q merely by adopting an amplitude dependence for the phase lags of the different tides. Unfortunately, this is really not sufficient since a tide of small amplitude will have a phase lag which increases when its peak is reinforcing the peak of the tide of major amplitude. This non-linear behavior cannot be treated in detail since very little is known about the response of the planets to tidal forces, except for the Earth. In our discussions we shall use the language of linear tidal theory, but we must keep in mind that our numbers are really only parametric fits to a non-linear problem.

There is one more assumption which is implied in this paper. It concerns the neglect of direct gravitational interactions between bodies in influencing the eccentricities of the satellites. Celestial mechanicians in particular, would consider this omission to be a very serious one over periods greater than a few thousand years. This is because their calculations will not guarantee the stability of satellite eccentricities, perturbed by direct gravitational interactions, for periods greater than these. To this objection one can only offer the belief that for well-space orbits, direct gravitational interactions alone will not endanger stability in eccentricity, even over ages comparable to those of the solar system.

7. The development of the orbit in time will be taken up in this section for use later on. From Jeffreys' paper[9] we have

$$\overline{\frac{da}{dt}} = \frac{18(G\,M_p)^{1/2}(G\,M_s)\rho_p R_p^4 \varepsilon_{op}}{a^{11/2}(19\mu_p + 2g_p\rho_p R_p)} \qquad (2\text{-}18)$$

In the absence of better information, Q is taken independent of frequency and amplitude in this section. This enables us to integrate the above equation which gives:

$$a = \left[\frac{117(G\,M_p)^{1/2}(G\,M)\rho_p R_p^4 \varepsilon_p\, t}{(19\mu_p + 2g_p\rho_p R_p)} + a_o^{13/2}\right]^{2/13} \text{ where } a = a \quad \text{at } t = 0.$$

If we set ε_ι $\iota = 1 \rightarrow 3$ equal to $\pm\varepsilon$ then

$$\overline{\frac{de}{dt}}\,p = \frac{e\alpha(G\,M_p)\;(G\,M_s)R_p^4\rho_p\varepsilon_p}{a^{13/2}(19\mu_p + 2g_p\rho_p R_p)}$$

$$\overline{\frac{de}{dt}}\,s = \frac{-e\;72(G\,M)\;R_s\rho_s\varepsilon_s}{a^{13/2}(19\mu_s + 2g_s\rho_s R_s)}$$

α is a numerical coefficient which depends on the sign of the various ε_ι.

In any case

$\overline{\frac{de}{da}} = \frac{\gamma e}{a}$ so $\left(\frac{e}{e_o}\right) = \left(\frac{a}{a_o}\right)^\gamma$ where $e = e_o$ when $a = a_o$. γ will be evaluated separately in each case in the following sections.

8. Numerical estimates for $\overline{\frac{de}{dt}}$ are made in this section for several different satellites.

i) The Earth-Moon System

Since we have more information in this case than in any

other, it will be investigated in greatest detail. Jeffreys[10] uses $\frac{19}{2g\rho R}=3$ for the Earth, with a homogeneous sphere model. If we use the same value of μ for the Moon as for the Earth, then we get

$$\frac{\overline{\frac{de}{dt}}_s}{\frac{de}{dt}_P} = -\frac{28}{19}\left(\frac{1738}{6378}\right)\left(\frac{4}{3}\right)^2\frac{Q_{op}}{Q_{os}} = -.7\,\frac{Q_{op}}{Q_{os}}$$ where Q_o is the value of Q

for the material when self-gravity is negligible, as discussed in section 4. If $Q_{os}=Q_{op}$ then the eccentricity of the Moon's orbit would be increasing. However, there are two considerations which tend to increase the importance of the tides on the Moon. In the first place, the rigidity of the Moon is likely to be smaller than that for the Earth since the high rigidity of the Earth is due to high pressures in the interior. Furthermore, the strains on the Moon are larger than those on the Earth by a factor of 4.9, for the value of rigidity given above. It is known that for the Earth Q decreases with the amplitude of strain[11] and this would probably also be the case for the Moon.

The above would be considerably altered if ocean tides were significant contributors to Q on the Earth, but recent evidence[12] seems to rule out this possibility. If we take

$\mu_s=\frac{\mu_p}{2}$ then $\dfrac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} = -1.4\,\frac{Q_{op}}{Q_{os}}$. We thus see that small changes in

the parameters μ and Q alter the sign of $\overline{\frac{de}{dt}}$ so that the

results for the Moon must be considered inconclusive.

If we determine the value of $\frac{1}{(19\mu_p + 2g_p\rho_p R_p)Q_p}$, for the Earth, from the present observed secular acceleration of the Moon, we can use this information to determine the evolution of the semi-major axis of the Moon. MacDonald has done this and claims that the data is consistent with having the Moon close to the Earth between one-half and one billion years ago.[13] Ignoring the difficulties which arise from such a situation, we can make the following observation in regard to eccentricity: if the rate of eccentricity change is dominated by tides on the Earth, then we get $\left(\frac{e}{e_0}\right) = \left(\frac{a}{a}\right)^{\frac{48}{19}}$ (2-20)

For $a_o = 10{,}000$ km., $a = 380{,}000$ km. and $e = .055$ we get $e_o = 5\times10^{-6}$ (Groves gets 2×10^{-6}) which seems very small, especially when compared with the eccentricities of other satellites in the solar system, for which planet tides produce negligible secular accelerations. On the other hand, if e has been decreasing and we assume its initial value was less than $e_o = .5$, we get the following: From the result $\left(\frac{e}{e_0}\right) = \left(\frac{a}{a}\right)^{\gamma}$ we get $-.6<\gamma<0$, so we see that $\left|\frac{de}{dt}s\right| < 1.25\left|\frac{de}{dt}p\right|$ and the two rates of change must have almost canceled each other. In any case, it seems likely from the preceding that whether the Moon's eccentricity is increasing or decreasing, the two tidal effects are close to being equal in magnitude. It is worth noting, in this context, that the Moon's eccentricity is higher than that of all other inner satellites in

the solar system.

ii) Mars

As explained in section 5, Phobos and Deimos are covered by case i), section 5 and $\overline{\frac{de}{dt}}_s \Big/ \overline{\frac{de}{dt}}_p \ll 1$.

a) For Phobos $\varepsilon_{op}, \varepsilon_{1p}$, and ε_{2p}, are all negative while ε_{3p} is positive. This tells us that $\overline{\frac{de}{dt}}_p$ is almost certainly negative. This agrees with the observed low eccentricity of .019 for Phobos. It is still necessary to show whether the tides on Mars could have appreciably altered the eccentricity of Phobos in the age of the solar system, which we take as four billion years. Using the same μ_p for Mars as is used by Jeffreys,[14] we have:

$\overline{\frac{de}{dt}}_p = -\frac{9.75\times10^{-15}e}{Q_p}$ sec^{-1} at the present time. Next, using Q=100, which is a typical value for low amplitude tides on the Earth, we get $\frac{1}{e}\overline{\frac{de}{dt}}_p = -9.75\times10^{-17}$ sec^{-1}. Since 4×10^9 years $= 1.2\times10^{17}$ sec we see that e could have been appreciably decreased by tides on Mars. It should be borne in mind that the semi-major axis of this satellite is decreasing, since the satellite's period is shorter than the Martian day. This means that $-\frac{1}{e}\frac{de}{dt}_p$ was smaller in the past than it is now. Using formulas (2-19) and (2-20), one could carry this analysis out to include the integration over the past four billion years.

b) For Deimos, we have ε_{1p} negative while ε_{op}, ε_{2p} and ε_{3p} are all positive. This assures us that $\overline{\frac{de}{dt}}_p$ is definitely

negative for Deimos. Again using Jeffreys' numbers we have, taking $Q_p = 100$: $\frac{1}{e}\overline{\frac{de}{dt}}_p = 4.5\ 10^{-20}\ \sec^{-1}$, so we see that in 4×10^9 years, or $1.2\ 10^{17}$ sec., the tidal forces will make a negligible change in e. For Deimos, the semi-major axis is increasing due to the tides but at such a rate that this effect may also be safely neglected.

Since the eccentricity of Deimos is .003, we are forced either to accept the conclusion that Deimos was formed with this initial eccentricity or to find some other process that might have decreased the eccentricity of this satellite. In the following paper one such mechanism is proposed. It involves the direct gravitational interaction of Phobos with Deimos coupled with the tidal forces on Phobos. It is shown there, that the rate of change of Deimos' eccentricity due to this process, could have been significant over a period of four billion years.

ii) Jupiter, Saturn, Uranus and Neptune

As discussed in section 6, the satellites of these planets are likely to be solid, while the state or states of the planets are uncertain.

For Jupiter V the planetary, as well as the satellite, tides decrease the satellite's eccentricity. These tidal effects are very likely to be significant and probably account for the satellite's low eccentricity of .003.

For all the other satellites of these planets, except

for the retrograde ones, the planetary tide increases the satellite's eccentricity. These satellite planet systems probably can be approximated by case ii) of section 5, if we assume μ's of the order of those of ice. If we write equation (2-15) in terms of Q_{os} and Q_{op}, we get:

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} = -\frac{28}{19}\,\frac{(2g_p\rho_pR_p)^2\,Q_{op}}{19\mu_p\cdot 19\mu_s\,Q_{os}}\left(\frac{R_s}{R_p}\right)$$

If, for simplicity, we consider the case of a planet and satellite of the same material, this becomes

$$\frac{\overline{\frac{de}{dt}}_s}{\overline{\frac{de}{dt}}_p} = -\frac{28}{19}\left(\frac{2g_p\rho_pR_p}{19\mu}\right)^2\frac{R_s}{R_p}$$ Since $\frac{2g_p\rho_pR_p}{19\mu_p}>1$, we see that

all satellites with $R_s \geq \frac{19R_p(19\mu)^2}{28(2g_p\rho_pR_p)^2}$ will have decreasing eccentricities, while those with $R_s < \frac{19R_p(19\mu)^2}{28\,(2g_p\rho_pR_p)^2}$ will have increasing eccentricities.

After this brief and very speculative discussion, we can only appeal to observation, which shows small eccentricities for the five inner satellites of Jupiter, the six inner satellites of Saturn, the four major satellites of Uranus and the inner satellite of Neptune. In all cases where the eccentricity is less than .01 we find: $-\frac{1}{e}\,\overline{\frac{de}{dt}}_s \geq 1.2\times10^{17}\,\mathrm{sec}^{-1}$ for reasonable values of Q's and μ's. This seems to indicate that tides raised on satellites are of great significance in the evolution of the eccentricities of these satellites.

An Explanation of the Frequent Occurrence of Near-Commensurate Mean Motions in the Solar System

Introduction

In this paper an explanation of the improbably large number of near-commensurate pairs of satellite mean motions is proposed. It is shown that special cases of near-commensurate mean motions are stable under tidal forces. Furthermore, at least four of the best illustrations of commensurabilities in the solar system have this stability. Finally, the significance of these stable configurations on the evolution of satellite systems is discussed.

1. The existence of near-commensurabilites among the mean motions of the satellites and planets in the solar system has been known for many years. The most famous of these commensurabilities involves the Jovian satellites Io, Europa and Ganymede. Within observational accuracy, the mean motions (n_1, n_2 and n_3 respectively) of these satellites obey the relation $n_1-3n_2-2n_3=0$. The motions of these satellites have been studied in great detail, first by Laplace, and subsequently by many other authors. In addition to this three-body commensurability, several cases

of near-commensurabilities between the mean motions of two satellites have also been known for quite some time. The motions of these pairs of satellites have also been intensively studied since they yield data from which a determination of the satellite masses can be made.

More recently, A. E. Roy and M. W. Ovenden[1,2] have examined the mean motions of pairs of planets and satellites in a new light. They considered the question of whether the observed number of near-commensurate pairs of mean motions in the solar system was too great to have arisen from a random distribution of mean motions. As this paper is intended to provide answers to several intriguing questions that they raised, we shall begin with a general discussion of the contents of their two papers.

In their first paper, the authors arrived at the conclusion that the preference for near-commensurate mean motions in the solar system is inconsistent with the assumption of a random distribution of mean motions for the planets and satellites. A sketch of their proof of this important result will be presented next.

Before we can prove anything, however, a sharper definition of near-commensurate mean motions must be given. Let n_1 and n_2 $(n_1 > n_2)$ be the mean motions of two bodies about a common center of force. If two integers, A_1 and A_2 exist, such that $\left|\frac{n_2}{n_1} - \frac{A_2}{A_1}\right| = \varepsilon$ where ε is a small

positive number, then these mean motions are said to be nearly commensurate. Since the ratio $\frac{n_2}{n_1}$ can always be approximated with arbitrary accuracy by the ratio of two integers, it is necessary to limit the size of the integers considered. In their paper, Roy and Ovenden arbitrarily set this limit for A_1 at seven. This restriction to small integers in no way limits the scope of the discussion. In fact, it can easily be shown from perturbation theory that the importance of near-commensurabilities decreases as their order increases. Using our definition of near-commensurability, we can assign two integers, A_1 and A_2, to every pair of mean motions, n_1 and n_2, whose ratio $\frac{n_2}{n_1} \leq \frac{1}{7}$. Since the smallest difference between adjacent fractions is $\frac{1}{6}-\frac{1}{7}=\frac{1}{42}$, there can be at most one pair of integers, A_1, A_2 for each pair of mean motions, n_1, n_2 such that $\left|\frac{n_2}{n_1} - \frac{A_2}{A_1}\right| = \varepsilon < \frac{1}{82} = .01190$. From A_1 and A_2, with $A_1 \leq 7$, we can form 17 fractions with valies from $\frac{1}{7}$ to 1. Thus, given $\varepsilon_0 \leq .01190$, the probability that a randomly chosen ratio in the range $\frac{1}{7}$ to 1 lies within ε_0 of some fraction $\frac{A_2}{A_1}$ is $P_{\varepsilon_0} = 17 \times 2\varepsilon_0 \times \frac{7}{6} = 39.67\varepsilon_0$ Roy and Ovenden considered 46 pairs of mean motions and compiled a table which compares $46P_{\varepsilon_0}$, for various $\varepsilon_0 \leq .01190$, with the observed number of pairs of mean motions for which a near-commensurability exists with $\varepsilon \leq \varepsilon_0$. This table, minus the control distribution data, is reproduced below.

ε_0	.0119	.0089	.0059	.0030	.0015
$46P_{\varepsilon_0}$	21.7	16.2	10.8	5.3	2.5
Observed number of pairs of mean motions with $\varepsilon \leq \varepsilon_0$	33	26	20	12	6

Table (3-1)

As Roy and Ovenden pointed out, there are two reasons why the observed number of near-commensurabilities, which are listed in their table, might be misleading. In the first place, if n_2 has a near-commensurability with both n_1 and n_3, then it may also be the case that n_3 is nearly commensurate with n_1. If this is so, then it is unclear whether the commensurability between n_1 and n_3 should be considered as an independent one. In their paper, Roy and Ovenden showed that this problem of "multiple counts" was likely to affect the number of independent observed commensurabilities listed in table (3-1), by 2 or 3 for ε_0=.0119 and even less for smaller ε_0. The second source of error arises from the nonuniform distribution of the ratios $\frac{n_2}{n_1}$ on the interval $\frac{1}{7}$ to 1. In fact, no ratio exists which is greater than .75. While it is difficult to correct accurately for this effect, the authors do show that it should not affect the expected number of near-commensurabilities, for a given ε_0, by a factor very different from .925.

In light of the preceding discussion, we see that the distribution of mean motions very definitely deviates from randomness but that it is difficult to say precisely how large this deviation is.

In their second paper, Roy and Ovenden prove that, "if n point-masses are acted upon by their mutual gravitational forces only, and at a certain epoch each radius vector from the (assumed stationary) center of mass of the system is perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch." The authors call this theorem the mirror theorem and the special configuration described above is called a mirror configuration. As a corollary of the mirror theorem, the authors prove a periodicity theorem which states that, "if n point-masses are moving under their mutual gravitational forces only, their orbits are periodic if, at two separate epochs, a mirror configuration occurs."

After proving the preceding theorems, Roy and Ovenden suggest that the frequent occurrence of mirror configurations will cause perturbations on the orbits to undergo frequent reversals so that the disturbances they generate cannot build up to magnitudes so large that they endanger the stability of the motion. For a definition of stability, the authors adapt Poincare's conditions, namely:

i) The heliocentric distance of any planet cannot increase or decrease without limit.
ii) The system repeatedly passes through the configuration it had at time t_o, say at times t_1, t_2, t_3, etc.
iii) Close encounters of any pairs of planets are ruled out.
(The conditions for satellite orbits are analogous).

Finally, Roy and Ovenden examine three of the best cases (i.e., those for which ε is smallest) of near-commensurabilities in the solar system. These include three pairs of satellite orbits in Saturn's system: Hyperion and Titan, Enceladus and Dione, and Mimas and Tethys. The values of $\frac{n_2}{n_1} - \frac{A_2}{A_1}$ for these satellite pairs are -0.000566, +0.000643 and -0.000784 respectively.

Observation provides the following remarkable results: Conjunctions of Enceladus and Dione always occur near the perisaturnium of Enceladus. For Titan and Hyperion, the conjunctions always occur near the aposaturnium of Hyperion. For Mimas and Tethys, the relation involves their nodes and the conjunctions of these two satellies occur near the mid-point between their two ascending nodes on Saturn's equator plane.

The examination of these satellite pairs thus reveals that they satisfy the mirror theorem, at least to a first approximation. The nature of this approximation and its significance will be the subject of the rest of this paper.

2. Before we can proceed to the discussion of the stability of these near-commensurate mean motions, we must outline classical perturbation theory.

We shall describe the orbit of a satellite of mass m, about a planet of mass M, by the following six elements:

a is the semi-major axis of the orbit.

e is the eccentricity of the orbit.

i is the inclination of the orbit to the planet's equator.

Ω is the longitude of the ascending node.

$\tilde{\omega}$ is the longitude of the perihelion.

$\tilde{\omega}=\Omega+\omega$ where ω is the angle between the ascending node and the perihelion.

ε is the mean longitude of the satellite at epoch.

To give the position in the orbit, we will use the mean longitude λ $\quad \lambda=\int_0^t n\,dt+\varepsilon$

$n=\sqrt{\frac{GM}{a^{3/2}}}$ for motion about a spherical planet. It is called the mean motion since $n=\frac{2\pi}{P}$ where P is the satellite's revolution period.

For unperturbed motion about a spherical planet, a, e, i, Ω, $\tilde{\omega}$, and ε are constants. If the motion is perturbed, however, then a, e, i, Ω, $\tilde{\omega}$, and ε will, in general, vary with time. If $V(\vec{r})$ is the total potential per unit mass acting on the satellite, then $R(\vec{r})$ is defined to be $V(\vec{r})-\frac{GM}{r}$ and is known as the disturbing function. In terms of the disturbing function, we can write the equations of motion

for a, e, i, Ω, $\tilde{\omega}$, and ϵ. In what follows, R is considered to be a function of a, e, i, Ω, $\tilde{\omega}$, and λ. For simplicity, we shall neglect powers beyond the second in the satellite's eccentricity and inclination. This allows us to write:[3]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\
\frac{de}{dt} &= \frac{-1}{na^2 e}\frac{\partial R}{\partial \tilde{\omega}} \\
\frac{di}{dt} &= \frac{-1}{na^2 i}\frac{\partial R}{\partial \Omega} \\
\frac{d\epsilon}{dt} &= \frac{2}{na}\frac{\partial R}{\partial a} \\
\frac{d\tilde{\omega}}{dt} &= \frac{1}{na^2 e}\frac{\partial R}{\partial e} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2 i}\frac{\partial R}{\partial i}
\end{aligned} \qquad (3\text{-}1)$$

Perturbations of the first order are obtained by treating a, e, i, and $n = \frac{\sqrt{G M}}{a^{3/2}}$ as constants on the right-hand sides of the perturbation equations, while the mean longitude, λ, is treated as a linear function of the time. We shall deviate from common practice, however, and allow for secular motion of $\tilde{\omega}$ and Ω when we treat the periodic terms in the disturbing function. The only restriction this places on our development, is that it forces us to treat the secular part of the disturbing function first.

The disturbing function for the action of a satellite with mass m′ on one with mass m is given by

$$R=Gm'\left(\frac{1}{\Delta}-\frac{xx'+yy'+zz'}{(r')^3}\right)$$

Coordinates are measured from the center of the planet. Primes refer to the disturbing satellite.

$$\Delta^2=(x-x')^2+(y-y')^2+(z-z')^2$$

It can be shown that R can be expanded in the following form:[4] $R=\Sigma F(a,\ a',\ e,\ e',\ i,\ i')\cos T$

where $T=[h\lambda+h'\lambda'+g\tilde{\omega}+g'\tilde{\omega}'+f\Omega+f'\Omega']$ (3-2)

The requirement of rotational invariance gives us the single restriction

$h+h'+g+g'+f+f'=0.$ (3-3)

The results obtained from first order perturbation theory are only approximate due to our treating a, e, and i as constants in the right-hand sides of the perturbation equations. If necessary, the calculations can be extended to higher order (the order is measured by the power of $\frac{m'}{M}$) perturbations. This is done by substituting the results of the first order calculations in the right-hand members of the perturbation equations for a, e, i, Ω, $\tilde{\omega}$, and ε. We shall need higher order perturbation theory when we discuss commensurabilities of more than two satellites. One result that we shall quote for later use is Poisson's theorem on the invariability of the semi-major axis. This theorem states that there is no secular term, due to gravitational interactions between satellites, in the expression for the semi-major axis, both in the first and second orders of perturbation theory.

Since we will apply perturbation theory in cases where near-commensurabilities exist, a brief summary of the effects of near-commensurabilities in perturbation theory will be given next. If we write $R=\sum C\cos T$, then the perturbation equations yield, upon integration:

$$\delta_1 a = \sum -\frac{C_1 \cos T}{(hn+h'n')} \qquad \delta_1\varpi = \sum +\frac{C_2 \sin T}{(hn+h'n')}$$

$$\delta_1 e = \sum -\frac{C_1' \cos T}{(hn+h'n')} \qquad \delta_1\Omega = \sum +\frac{C_2' \sin T}{(hn+h'n')}$$

$$\delta_1 i = \sum -\frac{C_1'' \cos T}{(hn+h'n')} \qquad \delta_1\varepsilon = \sum +\frac{C_2'' \sin T}{(hn+h'n')}$$

(Note δ_1 denotes first order perturbations).

Here, in the interest of simplicity, ϖ and Ω have been treated as constants in the right-hand members of the perturbation equations. From the definition of $\lambda = \int_0^t n\,dt + \varepsilon$, we see that

$$\delta_1\lambda = \int_0^t \delta_1 n\,dt +$$

$$= \sum\left\{-\frac{C_2''' \sin T}{(hn+h'n')^2} + \frac{C_2'' \sin T}{(hn+h'n')}\right\}$$

A near-commensurability means that one of the expressions, $h^*n+h'^*n'$, is very small compared to n or n'. From the expression above, we see that a near-commensurability implies an enhanced amplitude for perturbations with period $h^*n+h'^*n'$. Since only $\delta_1\lambda$ has the small divisor squared, we see that the principal effect of a near-commensurability will be observed in the mean longitude.

3. We are now in a position to investigate the stability of near-commensurate mean motions. Our considerations will find application to the two-body cases of Enceladus and Dione, Mimas and Tethys, and Hyperion and Titan, as well as to the three-body case of Io, Europa and Ganymede. Other possible examples of stable commensurabilities to which these results might also apply are mentioned in section 10.

As a start, we shall consider a planet surrounded by several satellites which move on well-spaced orbits of low inclination and eccentricity. We shall make the assumption that the tidal torques on these satellites have produced considerable evolution of their mean motions over a period comparable to the age of the solar system (which we take as four billion years).

Let us make the further assumption that the tidal evolution of the mean motions of each satellite is independent of the other satellites. This independent evolution of mean motions is implied (at least to second order perturbation theory) by Poisson's theorem on the invariability of the semi-major axis (see section 2). Since, in general, the mean motions of different satellites will evolve at different rates, the ratios of the mean motions of pairs of satellies will vary with time. In so doing they will occasionally pass through a low order commensurability. However, at any one time, the ratios of the

satellites' mean motions will exhibit a tendency for near-commensurability which is consistent with a random distribution of mean motions. Such a situation would certainly fail to explain the strong tendency for near-commensurate mean motions that is observed among the satellites of the solar system.

If, on the other hand, these near-commensurabilities were stable, then we could account for the large number of observed near-commensurabilities. Suppose, for example, that during the tidal evolution of mean motions, the ratio of the mean motions of two satellites approaches very close to the ratio of two small integers. If a near-commensurate motion of these two satellites exists, which is stable under further tidal evolution, then the satellites will remain in the near-commensurability rather than merely passing through it. However, the tidal torque on each of these satellites will not be affected by such a near-commensurability of the satellite mean motions. This being the case, in order for the further evolution of the satellite orbits to proceed without disrupting the near-commensurability of the mean motions, angular momentum must be secularly transferred between the satellites. At first sight this condition might appear to be a violation of Poisson's theorem on the invariability of the semi-major axis. However, the proof of this theorem involves the

assumption that the arguments of the periodic terms in the disturbing function are not constant. As we shall see later on, this is just the condition which occurs in the cases of stable commensurabilities.

We now see that the orbits of a pair of near-commensurate satellites will still evolve as the tides feed angular momentum from the planet's spin into the satellites' orbits. However, the satellites will share this angular momentum between them in just the correct proportion so that their mean motions remain near-commensurate.

The question of which near-commensurabilities are stable will be dealt with next. From the discussion of the previous paragraph we see that a necessary condition for the stability of a near-commensurability is that the direct gravitational forces between the satellites involved are strong enough to be able to distribute the angular momentum fed into the system by the tides in the manner necessary to maintain the commensurability relation. Application of this condition will enable us to place bounds on the tidal torques of some satellites. This, in turn, will imply upper bounds for the dissipation of the tides within the planets.

An examination of these direct gravitational forces reveals that they decrease rapidly as the order of the commensurability increases (see section 9). This accounts for the low orders of the observed near-commensurabilities (e.g.,

2 to 1, 4 to 3 etc.). When the direct gravitational forces are so weak that they cannot transfer angular momentum between the satellites at a sufficient rate, the two satellites' mean motions will evolve independently, each at a rate determined by the tidal torque on the satellite.

As discussed in section 1, the orbits of several pairs of the best examples of near-commensurabilities demonstrate remarkable regularities. Not only do these satellite pairs exhibit near-commensurabilities, they also show a relation between their conjunctions and one or more of their orbital elements. Since the conditions relevant to the various satellite pairs differ only in detail, we shall concentrate our attention on the system of Enceladus and Dione whenever an explicit example is called for.

Denoting the orbital elements of Dione by primes, we can state the observations in the form below (see section 2 for a definition of these elements).

$$2n' - n - \frac{d\bar{\varpi}}{dt} = 0 \quad \text{and}$$

$$2\lambda' - \lambda - \bar{\varpi} = V \text{ where } V \text{ oscillates about}$$

$0°$ with a small amplitude. (Actually, as we shall see later on, V should oscillate about an angle very close to, but not equal to $0°$). The second relation states that conjunctions of Enceladus and Dione always occur near the perisaturnium of Enceladus. Thus we see that this commensurability relation implies that the term in the disturbing

function with argument V can produce secular changes in the semi-major axis. (Actually, these secular changes only occur when V oscillates about an angle different from 0°). This invalidates the proof of Poisson's theorem as explained previously. It may also be noted that what we have here should not be considered as a near-commensurability of mean motions but rather as an exact commensurability involving the mean motions of Enceladus and Dione together with the motion of the perisaturnium of Enceladus. The fact that we have a near-commensurability of mean motions is just a consequence of the small size of $\frac{d\bar{\omega}}{dt}/n$.

Roy and Ovenden tried to show that these near-commensurate satellite pairs satisfy the hypotheses of their mirror theorem. In the approximation that the inclinations of Enceladus and Dione are neglected and that the eccentricity of Dione is taken as zero, we see that this is the case. Furthermore, even when the eccentricity of Dione is taken into account, it may still be argued that mirror configurations do occur. However, in the other cases of near-commensurability described by Roy and Ovenden, mirror configurations only occur in a first approximation to the actual orbits (e.g., when only one eccentricity is taken as non-zero or when both inclinations are considered to be equal, etc.). Finally, Roy and Ovenden remarked that when the mirror configuration was only approximately satisfied,

librations in the longitude of the satellites took place. The significance of these librations to their argument is unclear to the present author. However, it is well-known that these librations arise from terms in the disturbing function of very long period. For Enceladus and Dione, one such term has argument $W=2\lambda'-\lambda-\varpi'$. This argument can be rewritten as $W=2\lambda'-\lambda-\varpi+(\varpi-\varpi')\simeq(\varpi-\varpi')$. This is just the term which gives rise to the libration of period 3.89 years discussed by Roy and Ovenden in their second paper.

We have proposed that the near-commensurabilities are a consequence of the tidal evolution of satellite orbits. If this hypothesis is to be tenable, then the tidal evolution of the satellite systems involved must have been appreciable in the age of the solar system. Using the data tabulated by Jeffreys[5] we can express $\frac{d\xi}{dt}$, (ξ is defined by $n=n_o\xi^{-3}$ where n_o is the present mean motion of the satellite. Hence, $\frac{dn}{dt}=-3\frac{d\xi}{dt}$ at the present time) for the various satellites in the solar system. In doing this, we take a homogeneous sphere model for the planet. μ is its rigidity, Q its specific dissipation function, ρ its density, g its surface gravity and R its radius. The tabulation below includes all satellites, about planets having two or more satellites, for which

$$\frac{d\xi}{dt}\Big/(1+\frac{19\mu}{2g\rho R})Q\geq 10^{-17}\ \text{sec}^{-1}\ .$$

Satellite	$\frac{dn}{dt} = \frac{d\xi}{dt} \Big/ \left(1+\frac{19\mu}{2g\rho R}\right) Q$ sec^{-1}
Phobos	2.6×10^{-14}
Jupiter V	7.0×10^{-15}
Io	4.9×10^{-13}
Europa	1.6×10^{-14}
Ganymede	2.5×10^{-15}
Callisto	4.0×10^{-17}
Mimas	4.0×10^{-14}
Enceladus	1.6×10^{-14}
Tethys	3.2×10^{-14}
Dione	1.0×10^{-14}
Rhea	2.0×10^{-15}
Titan	6.9×10^{-16}
Ariel	2.3×10^{-15}

Table (3-2)

In our notation $\frac{1}{Q}$ is equivalent to $\sin 2\varepsilon$ in Jeffreys' notation. Also of importance is the fact that 4×10^{9} years 1.2×10^{17} seconds. Integrating Jeffreys' equation for $\frac{1}{n}\frac{dn}{dt}$, we find that if n is the present value of the mean motion of a satellite, then the mean motion of that satellite was many times larger T years ago, where $T=\frac{1}{13}\left(\frac{d\xi}{dt}\right)^{-1}$. $\frac{d\xi}{dt}$ is evaluated at the present time. (This only applies if $\frac{d\xi}{dt}>0$. If $\frac{d\xi}{dt}<0$, then n was smaller in the past). From

this formula we see that for satellites whose present values of $\frac{d\xi}{dt}$ are greater than 6.4×10^{-19} sec^{-1}, appreciable evolution of their mean motions has occurred in the age of the solar system.

From the discussion to follow, we shall see that the stable commensurability relations are between Mimas and Tethys, Enceladus and Dione, Titan and Hyperion, and Io, Europa and Ganymede. It is also likely that Io and Europa, Europa and Ganymede, and Ganymede and Callisto form stable two-body commensurabilities and that Dione, Rhea and Titan take part in a stable three-body commensurability.

From Table 2 we see that in all the above mentioned commensurabilities at least one of the satellites involved has $\frac{dn}{dt} \geq .69\times10^{-15}$ sec^{-1}. This indicates that even with $\frac{1}{Q\left(1+\frac{19\mu}{2g\rho R}\right)}$ as small as 10^{-4}, the tides would still have produced considerable changes in these systems in the past. For Jupiter $1+\frac{19\mu}{2g\rho R}$ cannot be much greater than 1 and it probably is near 1 for Saturn also. This implies that Q for Jupiter and Saturn is about 10^{4}.

If we turn the above argument around then we can ask the following question: What percentage of all satellites, for which $\frac{d\xi}{dt} \Big/ Q\left(1+\frac{19\mu}{2g\rho R}\right) \geq 10^{-14}$ sec^{-1} and for which another satellite exists with period differing by less than a factor of seven from its period, are involved in a commensurability relation? Inspection of Table 2 tells us that

every satellite, except Phobos, which satisfied the preceding conditions, is part of at least one definitely stable commensurability relation. Furthermore, it is no surprise that Phobos is not part of a stable commensurability with Deimos, the other satellite of Mars. The mutual gravitational interactions between Phobos and Deimos will be very weak since both satellites are extremely small (about 8 and 4 km. in radius, respectively). If we look next at the same group of satellites but consider those for which $\frac{d\xi}{dt} \Big/ Q\left(1+\frac{19\mu}{2g\rho R}\right) \geq .69\times10^{-15}$ sec^{-1} then we find that they all are involved in a commensurability relation except for Phobos, Jupiter V, Ariel and possibly Rhea.

The arguments presented in this section seem to imply a tidal origin for the commensurabilities and a Q of about 10^4 for Jupiter and Saturn. In the following sections of this paper other estimates of Q will be obtained.

4. In this section the details of a stability proof for near-commensurate satellites will be presented. As a start, we shall consider the following idealization of the system of Enceladus and Dione. Two satellites with masses m and m′ move in orbits about the same planet. The following restrictions are placed on their orbits:

$\frac{m}{M} \ll 1 \qquad \frac{m'}{M} \ll 1 \qquad e' = 0 \qquad i = i' = 0 \qquad a' > a.$

The question of whether a stable motion exists for these two bodies such that $2n' - n - \frac{d\overline{\tilde{\omega}}}{dt} = 0$, is the one we shall attempt to answer. $\frac{d\overline{\tilde{\omega}}}{dt}$ refers to the observed secular rate of change of $\tilde{\omega}$.

In line with the comment made in section 2, we shall not treat $\tilde{\omega}$ as a constant, but shall include all motions of $\tilde{\omega}$ due to secular terms in the disturbing function of m. Henceforth, we shall write $\tilde{\omega} = \tilde{\omega}_0 + \tilde{\omega}_1 t$. An important distinction must be made between $\frac{d\overline{\tilde{\omega}}}{dt}$ and $\tilde{\omega}_1$. $\frac{d\overline{\tilde{\omega}}}{dt}$ is the observed secular motion of $\tilde{\omega}$. $\tilde{\omega}_1$, however, is just that part of this secular motion which is produced by secular terms in m's disturbing function. Due to the commensurability relation, the periodic terms in the disturbing function, which have argument $2\lambda' - \lambda - \tilde{\omega}$, also produce secular motions of $\tilde{\omega}$. These are included in $\frac{d\overline{\tilde{\omega}}}{dt}$ but are not included in $\tilde{\omega}_1$.

In our stability proof, we shall neglect all periodic temrs in the disturbing function except the ones with argument $2\lambda' - \lambda - \tilde{\omega}$. It is easy to see why the terms with this argument might cause concern. By hypothesis, this argument has zero secular rate of change. Substituting these terms in the perturbation equations, one sees that they produce secular changes in the elements. It might be suspected that these secular changes will disrupt the relation $2n' - n - \frac{d\overline{\tilde{\omega}}}{dt} = 0$. However, we shall show that under certain

circumstances this does not happen.

Terms in the disturbing function whose arguments are integer multiples of $2\lambda'-\lambda-\tilde{\varpi}$ can be neglected, since their coefficients are smaller than those of the terms with argument $2\lambda'-\lambda-\tilde{\varpi}$ by e raised to the power of the absolute value of the corresponding integer multiple. All periodic terms whose arguments are not multiples of $2\lambda'-\lambda-\tilde{\varpi}$ produce only small magnitude, short period perturbations of the orbits since their amplitudes are not enhanced by integration (see section 2). Finally, the theorem stated in section 2 concerning the invariability of the semi-major axis tells us that we do not have to worry about secular terms in the disturbing function.

The stability proof runs as follows: Let $\Phi=2\lambda'-\lambda-\tilde{\varpi}$, then the terms in the disturbing functions with this argument are, to first order in e,

$$R_{\Phi} = -\frac{eGm'}{2a'}\left[\left(4+\alpha\frac{d}{d\alpha}\right)A_2(\alpha)\right]\cos\Phi$$

$$R'_{\Phi} = -\frac{eGm}{2a'}\left[\left(4+\alpha\frac{d}{d\alpha}\right)A_2(\alpha)\right]\cos\Phi$$

where $\alpha=\frac{a}{a'}$ and $A_2(\alpha)$ is a Laplace coefficient. A_2 can be expressed in terms of elliptic integrals. We shall need the value of $C(\alpha)=\left(4+\alpha\frac{d}{d\alpha}\right)A_2(\alpha)$ when we apply our results to the system of Enceladus and Dione, but at present we shall proceed without it.

Next, the first and second time derivatives of Φ will be evaluated.

$$\frac{d\Phi}{dt}=\frac{2d\lambda'}{dt}-\frac{d\lambda}{dt}-\frac{d\varpi}{dt}=2n'-n+\frac{2d\varepsilon'}{dt}-\frac{d\varepsilon}{dt}-\frac{d\varpi}{dt}$$

$$\frac{d\Phi}{dt}=2n'-n+2\varepsilon_1'-\varepsilon_1-\tilde{\omega}_1+\frac{2d\varepsilon_0'}{dt}-\frac{d\varepsilon_0}{dt}-\frac{d\tilde{\omega}_0}{dt}$$

In the above, we have set: $\varepsilon=\varepsilon_0+\varepsilon_1 t$, $\varepsilon'=\varepsilon_0'+\varepsilon_1' t$ and $\tilde{\omega}=\tilde{\omega}_0+\tilde{\omega}_1 t$. ε_1, ε_1' and $\tilde{\omega}_1$ are just the motions of ε,ε' and $\tilde{\omega}$ due to the secular terms in the disturbing function. Since we are restricting our analysis to first order perturbation theory,

$$\frac{d\varepsilon_1}{dt}=\frac{d\varepsilon_1'}{dt}=\frac{d\tilde{\omega}_1}{dt}=0$$

Using the perturbation equations (3-1), we can now write:

$$\frac{d\varepsilon_0}{dt}=ne\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)^2\frac{dC(\alpha)}{d\alpha}\cos\Phi$$

$$\frac{d\varepsilon_0'}{dt}=-n'e\left(\frac{m'}{M}\right)\left[C(\alpha)+\left(\frac{a}{a'}\right)\frac{dC(\alpha)}{d\alpha}\right]\cos\Phi \qquad (3\text{-}4)$$

$$\frac{d\tilde{\omega}_0}{dt}=\frac{n}{2e}\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)C(\alpha)\cos\Phi$$

Taking another time derivative of Φ we get:

$$\frac{d^2\Phi}{dt^2}=\frac{2dn'}{dt}-\frac{dn}{dt}-ne\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)^2\frac{dC(\alpha)}{d\alpha}\sin\Phi\frac{d\Phi}{dt}$$

$$-2n'e\left(\frac{m}{M}\right)\left[C(\alpha)+\left(\frac{a}{a'}\right)\frac{dC(\alpha)}{d\alpha}\right]\sin\Phi\frac{d\Phi}{dt}$$

$$+\frac{n}{2e}\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)C(\alpha)\sin\Phi\frac{d\Phi}{dt}$$

From $n^2a^3=GM$, we have $\frac{dn}{dt}=-\frac{3}{2}\left(\frac{n}{a}\right)\frac{da}{dt}$ (3-5)

Using this expression we can re-write the equation for $\frac{d^2\Phi}{dt^2}$ as follows:

$$\frac{d^2\Phi}{dt^2}=-\frac{3}{2}eC(\alpha)\left[4\left(\frac{m}{M}\right)n'^2+\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)n^2\right]\sin\Phi$$

$$\left[\begin{array}{l} ne\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)^2\frac{dC(\alpha)}{\alpha}+2n'e\left(\frac{m}{M}\right)\left\{C(\alpha)+\left(\frac{a}{a'}\right)\frac{dC(\alpha)}{d\alpha}\right\} \\ -\frac{n}{2e}\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)C(\alpha) \end{array}\right]\sin\Phi\frac{d\Phi}{dt} \qquad (3\text{-}6)$$

In some circumstances, the term containing $\sin\Phi\,\frac{d\Phi}{dt}$ may be neglected. If this term is dropped and if we use the fact that $C(\alpha)$ is positive (C will be computed at a later stage), then we see that the equation for Φ reduces to that of a simple pendulum. As is well-known, this equation can always be solved in terms of elliptic functions. However, for the case of Enceladus and Dione, a small amplitude approximation is justified. This yields:

$$\Phi=\Phi\sin\nu t \quad \text{where} \quad \nu^2=\frac{3}{2}eC(\alpha)\left[4\left(\frac{m}{M}\right)n'^2+\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)n^2\right] \qquad (3\text{-}7)$$

Hence, our result proves that this special case of a near-commensurability will not disrupt itself.

We are now in a position to formulate the condition under which the term containing $\sin\Phi\,\frac{d\Phi}{dt}$ may safely be neglected. For simplicity, we shall restrict ourselves to the case where m' is considerably bigger than m. In this case, the ratio of the neglected term to the one which was retained becomes:

$r=\left(\frac{1}{12ne^2}\right)\frac{d\Phi}{dt}$. If $\Phi=\Phi_0\sin\nu t$, then the largest value that r can take is $r^*=\frac{\nu\Phi_0}{12ne^2}=\frac{\Phi_0}{2}\sqrt{\frac{C(\alpha)}{3e^3}\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)}$ (3-8)

The next point to be dealt with is the stability of the relation $2n' - n - \frac{d\bar{\omega}}{dt}$ when tidal forces are present. In this connection, we will make use of Jeffreys' paper which implies that the rate of change of mean motion due to tidal forces is given by

$$-\frac{dn_T}{dt} = \frac{27n^2}{4}\left(\frac{m}{M}\right)\left(\frac{R}{a}\right)^5 \frac{2\varepsilon_0}{(1+19\mu/2g\rho R)} \qquad (3\text{-}9)$$

where the notation is that used in section 3. Rewriting equation (3-6) to include tidal effects (again we neglect the $\sin\Phi \frac{d\Phi}{dt}$ term), we get

$$\frac{d^2\Phi}{dt^2} = -\nu^2 \sin\Phi + \frac{2dn'_T}{dt} - \frac{dn_T}{dt} \qquad (3\text{-}10)$$

If the tidal effects are weak and if the small angle approximation can be used, this case gives

$$\Phi = \Phi_0 \sin\nu t + \frac{1}{\nu^2}\left(\frac{2dn'_T}{dt} - \frac{dn_T}{dt}\right) = \Phi \sin\nu t + \gamma \qquad (3\text{-}11)$$

so that the only effect produced by the tides is a phase shift γ in Φ. Hence, we have shown that weak tidal forces do not upset the relation $2n' - n - \frac{d\bar{\omega}}{dt}$. The remarkable fact is that this relation is maintained as the satellite system evolves under the action of the tides.

If, on the other hand, we had a case for which

$$\left|\frac{2dn'_T}{dt} - \frac{dn_T}{dt}\right| >> \nu^2, \text{ then } \Phi \simeq \left(\frac{2dn'_T}{dt} - \frac{dn_T}{dt}\right)\frac{t^2}{2} \qquad (3\text{-}12)$$

and the relation $2n' - n - \frac{d\bar{\omega}}{dt} = 0$ would be broken down by the tidal forces.

5. This section will contain a numerical application of the results of the last section to the satellite system of Enceladus and Dione.

The data is given below.

Enceladus	Dione
$P=\frac{2\pi}{n}=1.370218$ days	$P'=\frac{2\pi}{n'}=2.736916$ days
$e = .0045$	$e' = .0021$
$i = 0.0°$	$i' = 0.0°$
$\frac{d\varpi}{dt}=123.43°$ per year	$\frac{d\varpi'}{dt}=30.74°$ per year

1 $\alpha = \left(\frac{a}{a'}\right) = .631$

Table (3-3)

Using the recurrence relations for the Laplace coefficients, we arrive at

$$C(\alpha)=\frac{\alpha(5\alpha^2-2)b^{0}_{1/2}+(4+3\alpha^2-10\alpha^4)b^{1}_{1/2}}{3\alpha(1-\alpha^2)}$$

where $b^{0}_{1/2} = \frac{4}{\pi}\int_0^{\pi/2}\frac{d\theta}{(1-\alpha^2\sin^2\theta)^{1/2}}$

$$b^{1}_{1/2} = \frac{4}{\pi\alpha}\left[\int_0^{\pi/2}\frac{d\theta}{(1-\alpha^2\sin^2\theta)^{1/2}} - \int_0^{\pi/2}(1-\alpha^2\sin^2\theta)^{1/2}d\theta\right]$$

For $\alpha=.631$ we get $C(\alpha)=1.79$. This establishes that $C(\alpha)>0$, a fact which we have used before.

Calculations of r^* and γ can be easily carried out now. They yield

$$r^* < 8.45\times10^{-3}$$

$$\gamma = \frac{-2.44\ 10^{-2}}{\left(1+\frac{19\mu}{2g\rho R}\right)Q} \quad \text{radians} \tag{3-13}$$

From the value for r* we see that our neglect of the $\sin\Phi\frac{d\Phi}{dt}$ term was justified. The value of the phase shift appears to be quite small. Even if $1+\frac{19\mu}{2g\rho R}\approx 1$, it is probably too small to be observed since Q is likely to be greater than 100 for Saturn (see section 3).

6. The system of Titan and Hyperion is similar to that of Enceladus and Dione. If we let primes denote quantities related to Hyperion, then we have $4n'-3n-\frac{d\tilde{\omega}'}{dt}=0$ and the angle $\Phi=4\lambda'-3\lambda-\tilde{\omega}'$ oscillates about 180°. This motion can be shown to be stable under tidal action in exactly the same manner as before. Unfortunately, in this case, the phase shift due to tides is even smaller than that for Enceladus and Dione and offers no hope for direct observation. Finally, the term in the disturbing functions with argument $4\lambda'-3\lambda-\tilde{\omega}'$ is

$$R_\Phi=\frac{e'Gm'}{2a'}\left[\left(7+\alpha\frac{d}{d\alpha}\right)A_3(\alpha)\right]\cos\Phi$$

$$R'_\Phi=\frac{e'Gm}{2a'}\left[\left(7+\alpha\frac{d}{d\alpha}\right)A_3(\alpha)\right]\cos\Phi \tag{-14}$$

Since the coefficient of $\cos\Phi$ can be shown to be positive, in this case, Φ is stable about 180° rather than about 0° as was the case for Enceladus and Dione.

7. In this section, we shall treat the system of Mimas and Tethys. For this system, the commensurability relation involves the node instead of the perisaturnium. The conjunction of the two satellites oscillates about the midpoint between their two ascending nodes on Saturn's equator.

As an approximation to this system, we shall neglect e and e' and set $i=i'=\sqrt{\frac{i^2+i'^2}{2}}$ (primes refer to Tethys). The commensurability relation states that $\Phi=4\lambda'-2\lambda-\Omega'-\Omega$ oscillates about 0°. The terms in the disturbing functions with this argument can be shown to have the following forms:

$$R_{\Phi}=\frac{-i^2\,Gm\;B_3(\alpha)}{4a'}\cos\Phi$$

$$R_{\Phi}=\frac{-i^2\,Gm\;B_3(\alpha)}{4a'}\cos\Phi$$

where $B_3(\alpha)$ is a Laplace coefficient and is positive for this satellite system.

Neglecting the $\sin\Phi\frac{d\Phi}{dt}$ term, as in section 3, we get

$$\frac{d^2\Phi}{dt^2}=-\nu^2\sin\Phi \text{ where } \nu^2=3i^2\,B_3(\alpha)\left[4n'^2\left(\frac{m}{M}\right)+n^2\left(\frac{m'}{M}\right)\left(\frac{a}{a'}\right)\right]$$

For the system of Mimas and Tethys, a small angle approximation cannot be used since Φ oscillates about 0° with an amplitude of 95.3°. In this case, our pendulum equation must be solved in terms of elliptic integrals. The tidal forces produce a phase shift in Φ but again we can show that it is too small to be observed.

8. The case of Io, Europa and Ganymede was first discussed by Laplace and since then has been treated by many authors.[6] In this system, the commensurability relation only involves the mean motions of these three bodies. (They will be denoted by n_1, n_2 and n_3). $\lambda_1 - 3\lambda_2 - 2\lambda_3$ oscillates about 180° with a very small amplitude. The proof of the stability of this relation, under tidal forces, involves second order perturbation theory; otherwise, it goes through in exactly the same manner as the stability proofs in the two body cases. The stability for this case was known to Laplace.[7] In this case also, the phase shift in Φ is too small to be observed.

In the past, many authors have placed a lower limits on the Q of Jupiter. This was accomplished by noting that observations of Io (Jupiter I) gave no secular acceleration, to within the observational accuracy. Lower bounds for Q as high as 10^4 have been set by this method. It is interesting to see how the tidal stability of the commensurability relation changes the estimates of this lower bound. The amount of angular momentum which the tides transfer from the spin of Jupiter into orbital angular momentum of the satellites is unaffected by the commensurability. However, in order to maintain the relation $n_1 - 3n_2 - 2n_3 = 0$, this angular momentum must be shared among Jupiter's satellites in a special way. Three possibilities will be considered.

It is well-known, that in addition to the three-body commensurability just discussed, Jupiter's Galilean satellites have the following near two body commensurabilities: $2n_2-n_1\simeq 0$, $2n_3-n_2\simeq 0$ and $7n_4-3n_3\simeq 0$.[8] These may or may not be stable. (See section 9). If we assume the stability of the first two two-body commensurabilities and the three-body one, then the lower bound of Q must be decreased by a factor of 5. If we assume all the above commensurabilities are stable, then we must decrease this bound by a factor of 7.4. If only the three-body commensurability is stable, we again get a reduction factor of about 5.

9. The next topic to be discussed will be the evolution of these stable commensurability relations.

Except for the three-body case of Io, Europa and Ganymede, the commensurability relations we have considered have involved the pericenters and nodes of the satellites. From the stability equation (3-10) for Enceladus and Dione, we see that if e is below a certain value, then the commensurability relation is broken down by the tides. For Titan and Hyperion, a similar result holds for Hyperion's eccentricity, e'. The stability of the Mimas-Tethys commensurability would be destroyed if i became too small.

The inner satellites of planets tend to have small

eccentricities and inclinations. Since terms in the disturbing functions with argument $A_1\lambda_2 - A_2\lambda_1 - (A_1 - A_2)\varpi_1$ have at least $A_1 - A_2$ powers of e_1 in their coefficients, and those of argument $2(A_1\lambda_2 - A_2\lambda_1 - (A_1 - A_2)\Omega_1)$ have at least $2(A_1 - A_2)$ powers of i_1 in their coefficients, we see that it is not surprising that the commensurate relations we observe have $A_1 = A_2 + 1$. In this connection, we should note that, in general, these arguments can involve both pericenters and nodes, and that the only requirement on these arguments is that of rotational invariance, mentioned in equation (3-3).

Now that we understand why only commensurabilities of the form $A_1 = A_2 + k$, where k is a small integer, are stable under tidal forces, we can speculate about the past behavior of satellite systems. Appealing to the results of the previous paper we also know that the eccentricities of many satellites might have been considerably larger in the past than they are at present. A similar result has been proved for inclinations by Darwin.[9] This suggests that in the past, stable near-commensurabilities might have been much more common than at present, and of considerably greater variety (i.e., larger differences $A_1 - A_2$ were possible). Of these commensurabilities, all except those which involved the fewest powers of eccentricity and inclination would then have been disrupted by the tides. This would have occurred as the eccentricities and inclinations of these orbits decreased, also due to tidal action. Remnants of

these higher order commensurabilities may account for some of the near-commensurabilities which we now observe, but for which no stability relation, such as those discussed, exists.

Finally, there is still another way in which secular changes in eccentricity and inclination can be produced. This will be illustrated for the case of Enceladus and Dione.

Referring to equations (3-1), (3-4), (3-11) and (3-13), we have, for the rate of change of Enceladus' eccentricity due to the term in the disturbing function with argument Φ:

$$\frac{de}{dt}=\frac{Gm}{2a}\,C(\alpha)\sin\gamma$$

Since γ is negative, in this case we see that the eccentricity of Enceladus is being decreased by the action of Dione. This will eventually weaken the stability of the commensurability relation between Enceladus and Dione to a point where the tidal torque will disrupt it. Possibilities of this kind make it impossible to consider very seriously, the exact present values of the eccentricities of satellites in the contaxt of the previous paper alone. It is also possible, that a past commensurability between Phobos and Deimos could account for the low value of eccentricity for Deimos, as discussed in the previous paper.

Before we end this discussion, it should be mentioned that there is a limit beyond which these processes may not decrease the eccentricity of an orbit. That limit is given by the eccentricity which would be forced on the orbit by

direct gravitational interactions with other satellites and the sun.

10. In this section further areas for possible investigation are discussed.

In the first place, the stable commensurability relations which were discussed in this paper referred only to satellite systems and not to the planetary system. Tidal effects on the planet's orbits are too small to have any significance, even over ages comparable to that of the solar system. However, the stability proof discussed in this paper, would apply equally well to other phenomena which might produce secular changes in the semi-major axis of the planets. In particular, during the process of planet formation such forces would undoubtedly have existed in one form or another. It is then possible, that the planets might also have been involved in commensurability relations of the types discussed, and that their present distribution of mean motions is at least partially a reflection of these relations.

Secondly, other stable commensurability relations may exist. Possible candidates appear to be the two body cases of Io and Europa, Europa and Ganymede, and Ganymede and Callisto. For the first two of these pairs, Griffin[10] has remarked that the inner of the pair is near perijove and the

outer near apojove, whenever a conjunction takes place. However, due to the large perturbations present in these systems, a much more detailed stability proof is called for.

Finally, a possible three-body commensurability between Titan, Rhea and Dione is suggested. If we denote the elements relevant to these satellites by subscripts 3, 2, and 1 respectively, then the following relations may be noted:

$6n_3-5n_2+2n_1=000.081529$ degrees per day [11]

$\varpi_1+2\Omega_3=.0815\pm.001$ degrees per day [12]

Therefore, $6n_3-5n_2+2n_1-\varpi_1-2\Omega_3=000.0000\pm.0009$ degrees per day

We see that this relation holds to within the observational accuracy of six significant figures. If this is a stable commensurability relation, then a direct observation of the tidal phase shift should be possible.

LITERATURE CITED

CHAPTER I

1. Gray, A., A Treatise on Gyrostatics and Rotational Motion, New York: Dover Publications Inc., 1959, p. 207.

2. Woolard, E. W., A. J., 59 (1933), p. 33.

3. Brouwer, D. and Clemence, G., Methods of Celestial Mechanics, New York: Academic Press, 1961, p. 301.

4. Kozai, Y., A. J., 64 (1959).

5. Becker, R. A., Introduction to Theoretical Mechanics, New York: McGraw-Hill Book Co. Inc., 1954, p.253.

6. Brouwer and Clemence, op. cit., p. 125.

CHAPTER II

1. Darwin, G. H., Scientific Papers, Vol. II, New York: Cambridge University Press, 1908.

2. Groves, G. W., M.N.R.A.S., 121 (1960), p. 497.

3. Jeffreys, H., M.N.R.A.S., 122 (1961), p. 339.

4. Urey, H. C., Elsasser, W. M., and Rochester, M. G., Ap. J., 129 (1959), p. 842.

5. Jeffreys, H., The Earth, New York: Cambridge University Press, 1952.

6. Darwin, op. cit.

7. Jeffreys, op. cit., 122 (1961).

8. Munk, W. H. and MacDonald, G., The Rotation of the Earth, New York: Cambridge University Press, 1960.

9. Jeffreys, H., M.N.R.A.S., 117 (1957), p. 585.

10. Ibid.

11. MacDonald, G. (private communication).

12. Ibid.

13. Ibid.

14. Jeffreys, op. cit., 117 (1957).

CHAPTER III

1. Roy, A. E. and Ovenden, M. W., *M.N.R.A.S.*, 114 (1954), p. 232.

2. Roy, A. E. and Ovenden, M. W., *M.N.R.A.S.*, 115 (1955), p. 296.

3. Brouwer, D. and Clemence, G. *Methods of Celestial Mechanics*, New York: Academic Press, 1951, p. 319.

4. Plummer, H. C., *An Introductory Treatise on Dynamical Astronomy*, New York: Dover Publications, Inc., 1960, p. 176.

5. Jeffreys, H., *M.N.R.A.S.*, 117 (1957), p. 585.

6. Hagihara, Y., The Stability of the Solar System. Chapter 4 of *Planets and Satellites*, ed. G. Kuiper and B. Middlehurst, Chicago: University of Chicago Press, 1961, p. 112.

7. Laplace, P. S., *Méchanique Céleste*, Boston: Hillard, Gray, Little and Wilkins, 1829-39, Vols. I and IV.

8. Roy and Ovenden, *op. cit.*, 114 (1954).

9. Darwin, G. H., *Scientific Papers*, Vol. II, New York: Cambridge University Press, 1908.

10. Griffin, F. L., *Periodic Orbits*, ed. F. R. Moulton, Washington: Carnegie Institution of Washington Publication, No. 161, 1920, p. 425.

11. *Connaissance des Temps*, Paris: Bureau des Longitudes, 1954, Vol. XXX.

12. Brouwer, D. and Clemence, G., Orbits and Masses of Planets and Satellites. Chapter 3 of *Planets and Satellites*, ed. G. Kuiper and B. Middlehurst, Chicago: University of Chicago Press, 1961, p. 73.